The History of Ancient Indian Mathematics

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PREFACE

A few years ago, in 1958, I published in the *Mysore University Publications Series* a book in Kannada, entitled "The History of Mathematics". The first part of this work was devoted to a concise history of world mathematics, while the second part was devoted to an elementary exposition of the history of ancient Indian Mathematics. The book, being written in Kannada, was intended for the average lay man of the local area, and hence it was out of place to go into details with full mathematical explanations. Since then, I had the ambition to write a suitable book on the History of Indian Mathematics in English, with adequate mathematical notes and explanations. Due to various pre-occupations and other circumstances, the book could not be completed till now. I now venture to place this book before the mathematical world.

The only available book on the History of Indian Mathematics is the book "History of Hindu Mathematics" by B. B. Datta and A. N. Singh. This was originally published in two volumes. The two volumes have been published together recently by the *Asia Publishing House*, Bombay. The plan of the present book differs substantially from that of Datta and Singh. Datta and Singh's book is in the form of a "Source-Book" where in different topics are treated in the light of contributions to them by different authors at different times. The present work is *chronological* in its plan. Starting from the Vedic period, we begin with an account of the mathematical work contained in the *Sulva-Sutras*, in the ancient Jaina religious works, and in what is known as the Bakhshali manuscript. This brings us up to Arya Bhata (499 A.D.). We then set forth the main contributions made by Arya Bhata, Brahmagupta, Mahavira and Bhaskara in chronological order. Adequate mathematical explanations and critical remarks are included. Two important topics, the *Kuttaka* and the *Varga-Prakriti* figure in the mathematical works of more than one author. Hence these two topics are dealt with separately, in chapters IX and X, where in the relevant methods are explained, and the modifications suggested by

the different authors are discussed. The last chapter of the book is devoted to post-Bhaskara mathematics, which may as well be called Kerala mathematics, and is based on an article by C. T. Rajagopal and Mukunda Marar published in Vol. 20 (1944) of the *Journal of the Royal Asiatic Society* (Bombay Branch).

Vol. I of the first part of Datta and Singh's book gives an elaborate account of the origin of the Indian numerals. This has not been treated in the present work, as Datta and Singh's book gives all necessary information on this subject.

In compiling this work, I have been indebted to many sources. Datta and Singh's book mentioned above has been invaluable, and has been the main source of inspiration. The chapter on the *Sulva Sutras* has been largely based on B. B. Datta's "The Science of the Sulba". Colebrooke's translation with notes of Brahmagupta's *Brahma Sphuta Siddhanta* and Bhaskara's *Bijaganita*, as also H. C. Banerjee's translation of the *Lilavati* have been quite useful. Besides these, many original articles by various authors in different mathematical periodicals have been immensely made use of, and broadly speaking, this book has collected and presented in suitable form the mathematical matter found in these articles.

The book has been interspersed with original slokas or quotations from the Sanskrit texts. Direct translations or explanations of these slokas accompany the slokas.

So far, no chronological account of the contributions to mathematics by the ancient Indians has been available. The author hopes that this book satisfies a long-felt want. The subject-matter is of course confined to pure mathematics. A critical interpretation and estimate of the contributions of ancient Indians in the field of astronomy is a subject which is yet to be properly understood and written..

C. N. Srinivasiengar

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Chapter I

THE DECIMAL SYSTEM OF NUMERATION

1. The history of the remotest civilization is always shrouded in mystery and surrounded by controversy. The conclusions of a scientist can be challenged only on the grounds of accuracy of his experimental work and on the validity of his theory. But the conclusions of a historian can always be challenged more seriously on the grounds of insufficient data and knowledge, not to speak of his personal or national affections and prejudices. Civilization is spread over distant lands and languages. No one can vouchsafe that he has had access to all the extant literature in the different languages, that he has really mastered the meaning and the spirit of the different texts, and that he has not been misled by the translations or interpretations given by his predecessors. The last word on the history of ancient civilization will never be said.

A mathematician who writes the history of his science is by the very nature of human tendencies first a historian, and then a mathematician. Examples are not wanting when he has become unfortunately a politician. The conclusions as regards dates and origin of the achievements of ancient Indian mathematicians, and even of the extent of their knowledge, as made by some of the Western historians have been severely criticised by Indian scholars as unfair and prejudiced. The writings by Indian scholars of their own history have been comparatively few, and have not received the attention that they deserve at the hands of the historians of the West. A similar state of affairs exists in all probability as regards the achievements of other ancient civilizations like those of China, Chaldea and Egypt.

Under such circumstances, the best thing for a historian is to set the facts as he finds them or as he understands them, and allow the people of the world to interpret and assess these facts according to their own understanding of the subject.

2. The most fundamental contribution of ancient India to the progress of civilization is the invention of what is called the decimal system of numeration including the invention of the number "Zero". The characteristic feature of this system is the usage of nine digits and a symbol for zero to denote all integral numbers, by assigning a place-value to the digits. This system is so simple and is now learnt by children of tender age all over the world, thàt the profundity of its invention is easily lost sight of. The profundity of its invention is understood only when one realises the difficulty of making progress in arithmetic with other systems, such as the Greek system of numerals I, II, III, X, C, L etc. The Greek method of representing numbers by geometrical segments, and the slow progress of mathematics in the West before the advent of the Indian system to the West are sufficient illustrations of the handicap of the people who were not acquainted with the decimal system.

During the earlier decades of this century as well as in the last century, attempts were made to credit this invention wholly or in part to the Arabs. But mathematicians are now generally convinced that the invention is entirely the work of the ancient Hindus, and that the Arabs were the people who carried this invention to the states of Africa and of Europe.

3. We shall set forth such facts as are available in the ancient Sanskrit texts which have a bearing on the subject-matter of the decimal system of numeration. In the *Yajurveda Samhita* (*Vājasaneyee*), xvii, 2, the following list of numeral denominations is given : Eka (उक), Dasa (दश), Shata (शत), Sahasra (सहस्र), Ayuta (अयुत), Niyuta (नियुत), Prayuta (प्रयुत), Arbuda (अर्बुद), Nyarbuda (न्यर्बुद), Samudra (समुद्र), Madhya (मध्य), Anta (अंत), Parārdha (परार्ध). The same list occurs at two places in the *Taittireeya Samhita*, iv, 40.11.4 and vii, 2.20.1. The *Maitreyani Samhita*, ii, 8.14 and other sections of the Vedas give slight variations. Gaps are possible in the above list, for according to later Sanskrit literature, Arbuda means ten crores, i.e. 10^8 and not 10^7 as the above list would suggest.

We next refer to the *Valmiki Ramayana*, vi, 28. In this section, a spy of the demon king Rāvana narrates to his king the exact strength of his rival's (Rāma) army. Five persons from Rāvana's side, viz. his brother Vibhīsana and his four ministers, have deserted

into the opponent's camp. The spy also explains the numeration system employed. The following is his statement :

शतं शतसहस्राणां कोटिमाहुर्मनीषिण:
शतं कोटिसहस्राणां शङ्ख इत्यभिधीयते
शतं शङ्खसहस्राणां महाशङ्ख इतिस्मृतम्
महाशङ्ख सहस्राणां शतं वृंद इतिस्मृतम्
शतं वृंद सहस्राणां महावृंद इतिस्मृतम्
महावृंद सहस्राणां शतं पद्ममिवोच्यते
शतं पद्मसहस्राणां महापद्ममितिस्मृतम्
महापद्मसहस्राणां शतं खर्वं मिवोच्यते
शतं खर्वं सहस्राणां महाखर्वमितिस्मृतम्
महाखर्वं सहस्राणां समुद्रमभिधीयते
शतमोघ सहस्राणां महौघ इति विश्रुत:

This means the following enumeration :

100 lakhs = 1 Koti (crore) = 10^7. 10^5 Koti = 1 Sankha = 10^{12}. 10^5 Sankha = 1 Mahasankha = 10^{17}. 10^5 Mahasankhas = 1 Vrinda = 10^{22}. 10^5 Vrindas = 1 Mahavrinda = 10^{27}. 10^5 Mahavrindas = 1 Padma = 10^{32}. 10^5 Padmas = 1 Maha Padma = 10^{37}. 10^5 Maha Padmas = 1 Kharva = 10^{42}. 10^5 Kharvas = 1 Maha Kharva = 10^{47}. 10^3 Mahakharvas = 1 Samudra = 10^{50}. 10^5 Samudra (or Ogha) = 1 Mahaugha = 10^{55}. Rama's army is then estimated as 1000 Koti + 100 Sankhas + 1000 Maha Sankhas + 100 Vrinda + 1000 Maha Vrinda + 100 Padma + 1000 Mahapadmas + 100 Kharvas + 100 Samudras + 100 Mahaughas + 1 Koti Mahaughas + Vibhisana and his four ministers.
$= 10^{10} + 10^{14} + 10^{20} + 10^{24} + 10^{30} + 10^{34} + 10^{40} + 10^{44} + 10^{52} + 10^{57} + 10^{62} + 5$.

Ravana's army, according to Tāra (Vali's wife) is one hundred thousand crores + 36 ten thousands + one hundred thousand = $10^{12} + 36.10^4 + 10^5$.

The *Vedas* (parts of which are the *Samhitas*) are the scriptures of the Hindus. The *Ramayana* is a historical novel, with a religious and spiritual back-ground. One will not expect in these any rules or examples on arithmetic or geometry. The *Ramayana* depicts a very advanced and cultured civilization, and in the course of this big

novel, one finds numerous references to astronomy, public administration, engineering, textiles and music. It is but fair to agree that a nation with this civilization and which was using the above numeral system knew also how to handle the associated arithmetic.

As regards the age of these works, we simply state the following. Quoting from Datta and Singh, *History of Hindu Mathematics* (Part I).

"The Arab historian, Abul Hasan Al-Masudi (943), writes: 'A congress of sages at the command of the creator Brahma invented the nine figures, and also their (the Hindus') astronomy and other sciences'."

The Hindu scriptures give a similar though slightly different version of the origin of the Vedas themselves. The *Ramayana* comes much later. According to the Hindus, time is reckoned in terms of four Yugas, the lengths of the first three Yugas being respectively 4, 3, 2 times that of the Kali Yuga (कलि युग) whose duration is 432,000 years. We have just passed about 5000 years of the last, viz. Kali Yuga, which was preceded by Dwāpara (द्वापर) whose duration was 864,000 years. Dwāpara was preceded by Treta (त्रेता), and the *Ramayana* refers to an event that took place in the Treta period. This statement of dates based on orthodox scriptures differs so widely from the estimates given by modern savants, and we content ourselves with these remarks.

4. Writers on the Mohenjo Daro and Harappa civilizations which may be of the period 3000 B. C., or so, briefly refer to the decimal system of numeration found in those excavations. But they have not thrown much light on the subject, but the written documents, seals and inscriptions of this period as also the pottery belonging to the Megalithic (1500 B. C.) and Neolithic (6000—3000 B. C.) ages, preserved in the Madras Museum give the earliest available examples and specimens of writing in India. Apart from their own interest on this score, they are able to refute the theory that the ancient Indian script or scripts owe their origin to semitic or Phoenician sources.

Some further facts relating to the decimal system will be of interest. These will be of no use in establishing the antiquity of the system, except by way of stating that the system was in vogue on those dates.

The Jaina religious works, dating from 500 B. C. to about 100 B. C. use large numbers in the decimal system. No nation or community found the need for such huge numbers as have been employed by the Jainas and Buddhists. A subsequent chapter deals with Jaina mathematics. In the Buddhistic work *Lalita Vistara* of the first century B. C., Buddha (Bodhi Satva) enumerates to a mathematician Arjuna the system of numerals in multiples of 100, starting from *Koti* (10^7) up to 10^{53}.

Paleographic collections from the ancient Hindu colonies of the Far East include three inscriptions of Sri Vijaya which mention their dates as 605, 606 and 608 of the *Salivahana Saka* (Adding 78, we get the corresponding Christian date) in numeral figures. Two of these inscriptions were found at Palembang in Sumatra, and the third in the island of Banka. Another inscription giving the *Salivahana* date 605 was found at Sambor in Cambodia.

In an article in the *American Mathematical Monthly**, Prof. G. B. Halstead proves that the zero existed in India at the time of Pingala's work *Chhandas-Sutra* (छंदसूत्र)—a work on prosody—before 200 B. C.

5. The importance of the decimal system of numeration can best be appreciated in the words of other mathematicians. Laplace (1749—1827), one of the greatest mathematicians for all time writes : "The idea of expressing all quantities by nine figures (or digits) whereby is imparted to them both an absolute value and one by position is so simple that this very simplicity is the reason for our not being sufficiently aware how much admiration it deserves." Prof. Halstead† remarks, "The importance of the creation of the zero mark can never be exaggerated. This giving to airy nothing, not merely a local habitation and a name, a picture, a symbol but helpful power, is the characteristic of the Hindu race whence it sprang. It is like coining the *Nirvana* (निर्वाण) into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power."

* Vol. 33(1926), 449-54.

† *On the foundation and technique of Arithmetic*, Chicago (1912), p. 20.

CHAPTER II

THE SULVA SUTRAS

6. In many of the ancient countries, the development of mathematics was necessitated on account of religious practices and observances. These required an accurate calculation of the times of certain festivals and of the times auspicious for the performance of certain sacrifices or acts of worship. They also required a correct knowledge of the times of rising and setting of the sun and the moon, and of the occurrences of solar and lunar eclipses. All these meant a good knowledge of astronomy which in turn meant an accurate knowledge of arithmetic, plane and spherical geometry and trigonometry, and possibly also the construction of simple astronomical instruments.

In the Vedic religion, every house-hold man (i.e. barring the *Sanyasis* who would concentrate on meditation uninterruptedly for years) had to do certain acts of worship every day. It would be sinful if he neglected them. For purposes of worship, he would constantly maintain in his house three types of Agnis (अग्नि) or fires sheltering them in certain altars of special design. The Agnis were called *Dakshina*, *Gārhapatya* and *Ahavaneeya* (दक्षिण, गार्हपत्य, आहवनीय). The required altars had to be constructed with great care so as to conform to certain specific shapes and areas. The altar for the *Gārhapatya* Agni was square in one system, and circular in another system. The altars for the Ahavaneeya Agni and the Dakshina Agni were respectively square and semi-circular. The unit of length employed was the *Vyam* or *Vyayam* (व्याम, व्यायाम)· which was about 96 inches. Possibly this represented the height of the average man in those days, for the word *Purusha* (पुरुष) (=man) is also used for this length. The area of the altar had to be exactly one square Vyam, and the altar had to be constructed as accurately as was possible so as to conform to the rules.

7. While the above Agnis were to be used for the daily or routine *Pujas* or acts of worship, there were more elaborate sacrifices or Pujas for attaining certain cherished objects or wants. They were called *Kamyagnis* (काम्याग्नि). An act of worship done with a specific worldly desire is an inferior form of worship spiritually, but was popular with the Kings. Congregations of sages could also do this for the well-being of the entire community. Tne sacrificial altars for these *Kamyagnis* required more complicated constructions, involving combinations of rectangles, triangles and trapezia. The more elaborate sacrifices also required during the progress of a sacrifice the transference of the Agni to another altar either of the same shape or of a different shape, whose area bore a specific simple ratio to that of the original altar.

It will be clear that these processes require a clear knowledge of the properties of triangles, rectangles and squares, properties of similar figures, and a solution of the problem of 'squaring the circle' and its converse, 'circling the square' (i.e. to construct a square equal in area to a given circle, and vice versa).

8. These sacrifices or acts of worship are traceable to the remotest times. Several references to them are available in the *Rig Veda Samhita*. The science of the constructions of the altars takes a more specific form in the *Taittireeya Samhita* (तैत्तिरीय संहिता) and the *Taittireeya Brahmana* (तैत्तिरीय ब्राह्मण). Mention may also be made of the provision of these altars which Rama observes when he enters the hermitage of the sage Agasthya, and in Rama's own hermitages both at Chitrakoot and Panchavati.

In the hereditory handing down of instructions from father to son, and from the preceptor to the disciple so characteristic of the Hindu tradition in the past, the need for setting out the instructions in a written form was only slowly felt. In this way were written the several *Sulva Sutras* (शुल्वसूत्र) which were to be treated as adjuncts or appendices to the corresponding scriptural texts known as the *Srauta Sutras* (श्रौतसूत्र). The root meaning of the word *Sulv* is to measure, and in due course the word came to mean the rope or cord. Geometry in ancient India was for long known by the name Sulva or Rajju (रज्जु=rope). The name *Rekha Ganita* (रेखा गणित) is of later origin.

Only seven of the Sulva Sutras are known at present. They are known by the names *Bodhayana*, *Apasthamba*, *Katyayana*, *Manava*. *Maitrayana*, *Varaha* and *Vadhūla* after the names of the Rishis or sages who wrote them. The *Katyayana Sutra* belongs to the section of the Vedas called *Sukla Yajurveda* (शुक्ल यजुर्वेद) while all the rest belong to *Krishna Yajurveda*. The *Bodhayana*, *Apasthamba** and *Katyayana* Sulvas are of importance from the mathematical point of view.

The dates of these Sulva-Sutras have been estimated to be between 800 B. C. and 500 B. C. There is no knowledge about the existence of any Sulvas prior to these seven Sutras. It must be emphasized that the writers of the Sulva Sutras only wrote down and codified the rules for the constructions of the altars, that were in vogue from ancient times. They were not the persons who specified and directed the rules for the constructions of the altars.

9. The Sulvas explain a large number of simple geometrical constructions—constructions of squares, rectangles, parallelograms and trapezia. These and others involve the following theorems :

(1) The diagonal of a rectangle divides it into two equal parts.

(2) The diagonals of a rectangle bisect each other and the opposite areas are equal.

(3) The perpendicular through the vertex of an isoceles triangle on the base divides the triangle into equal halves.

(4) A rectangle and a parallelogram on the same base and between the same parallels are equal in area.

(5) The diagonals of a rhombus bisect each other at right angles.

(6) The famous theorem known after the name of Pythagoras.

(7) Properties of similar rectilinear figures.

These cover roughly the first two books and the sixth book of Euclid. How these theorems were actually obtained is a matter for which no definite answer is available. We all know that Euclid's geometry is based upon certain axioms and postulates, and the proofs involve a strict logical application of these. The logical methods of Greek geometry are certainly not discernible in Hindu

* Persons conversant with Sanskrit will find a neatly edited text of this in *Apasthamba Sulva-Sutras*, by D. Srinivasachar and S. Narasimhachar, *Mysore Oriental Library Publications*, No. 73 (1931).

geometry. No book on Hindu mathematics explains the system of axioms and postulates assumed, and this itself should go some way in refuting the concocted claim that Hindu mathematics is borrowed from the Greeks. At the same time, it may not be correct to conclude that the above theorems were asserted as a matter of experience and measurement. The people who could make out and solve complicated problems of arithmetic, algebra and spherical trigonometry should be credited with some amount of logic in their work. The Sulvas are not formal mathematical treatises. They are only adjuncts to certain religious works. The question has to end with these remarks.

10. The Theorem of Pythagoras occurs in the Sulvas in the following form :

The diagonal of a rectangle gives an area equal to the sum of the areas given by its length and breadth.

The proof of the Theorem might possibly have been obtained on the following lines :

Amongst the Vedic altars is one called the *Chaturashra Shyena Chit* (चतुरश्र श्येन चित्), whose shape is given below :

We have

$$AC^2 = ACEF$$
$$= ABCD + CDEG$$
$$= AB^2 + BC^2.$$

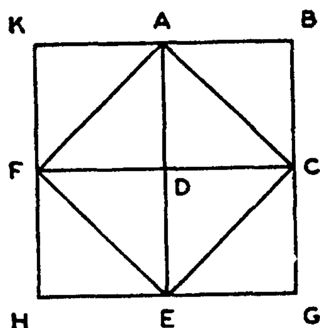


FIG. 1.

This figure corresponds to a particular case only, viz. $AB = BC$ i.e. the two sides of the rectangle are equal.

In Fig. 2, points are taken on the sides of the square $ABCD$, such that

$$BF = BK = CG = CL = DH = AE.$$

Now,

$$\text{sq. } ABCD = \text{sq. on } DL + \text{sq. on } BF + 4\triangle AEF$$
$$= AF^2 + AE^2 + 4\triangle AEF$$

EFGH is also a square, and

$$\text{sq. } ABCD = EF^2 + 4\triangle AEF$$

$$\therefore \quad AF^2 + AE^2 = EF^2.$$

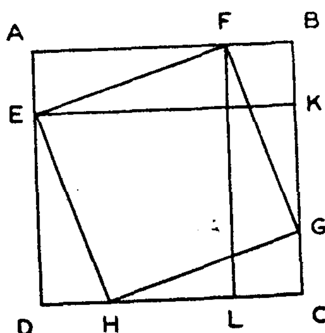


FIG. 2.

Figures of the type of Fig. 2 occur in the geometrical constructions discussed in the Sulvas, as will be seen below, and hence it is possible that the proof of the theorem of Pythagoras might have been obtained by the above method.

Pythagoras is believed to have lived from about 572 B. C. to 501 B. C., and the Sulvas are therefore prior to him. Pythagoras was therefore not the first person to enunciate the famous theorem named after him, and there is no evidence to show that he gave any proof. It is said that his proof was on the same lines as the above, and that another proof was the proof by using a right-angled triangle and the altitude from the vertex. The German historian Hankel is of the opinion* that these proofs savour the Indian style rather than the Greek style. The well-known proof given by Euclid may be his own. The theorem itself was known and widely used from very early times in Egypt, India and China.

11. There is another point of interest. The construction of a right-angled triangle whose sides are all integers is a problem of interest. In other words, integers a, b, c should be found such that $a^2+b^2=c^2$. The general solution of this is given by $a=m^2-n^2$, $b=2mn$, $c=m^2+n^2$ where m and n are any integers. This was first given by Brahmagupta (628 A. D.). Pythagoras gives the solution $2n+1$, $2n^2+2n$, $2n^2+2n+1$. This is not quite a general solution, since the difference between the last two sides is 1. Of course if a, b, c is a solution, ka, kb, kc is also a solution, but it need not be considered as a new solution. No general solution of the problem is given in the Sulvas, but a few examples are available where two sides do not differ by unity. The independent examples found in

* *Geschichte*, Leipzig (1874), p. 98.

the Sulvas are the solutions (3, 4, 5), (5, 12, 13), (7, 24, 25), (8, 15, 17) and (12, 35, 37). Several solutions equivalent to the first two are available. Thus the *Soutramani* (सौत्रामणि) altar uses a right-angled triangle with sides ($5\sqrt{3}$, $12\sqrt{3}$, $13\sqrt{3}$) and the Ashwa Medha (अश्वमेध) altar uses a triangle with sides ($15\sqrt{2}$, $36\sqrt{2}$, $39\sqrt{2}$). The irrational numbers evidently arise on account of the need for constructing an altar whose area is a multiple of the area of another altar of the same shape. Thus the dimensions of the *Soutramani* altar give a triangle whose area is three times that of the triangle (5, 12, 13). There are also big altars with dimensions (72, 96, 120) and (40, 96, 104). Apart from the use of irrational numbers, these figures unmistakably point to a clear knowledge of the properties of similar triangles, and the different examples cannot be the result of mere measurement and experience.

12. We shall now explain a few geometrical constructions contained in the Sulvas.

(a) To construct a square n times in area to a given square.

If a is a side of the given square, construct an isosceles triangle with base $(n-1)a$ and sides $\frac{1}{2}(n+1)a$. The square on its altitude is then the required square (*Katyayana Sulva*), for

$$\text{sq. on alt.} = \tfrac{1}{4}a^2(n+1)^2 - \tfrac{1}{4}a^2(n-1)^2 = na^2.$$

This therefore provides a simple geometrical construction for finding the square root of a given number n. The method should be contrasted with the method in Euclid.

(b) To construct a square of area equal to the sum of two given squares.

Along the sides of the bigger square $ABCD$, cut off $AE = DF =$ a side of the smaller square. Then the square on AF is the required square, for

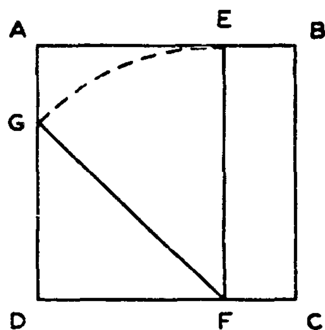


FIG. 3.

$$AF^2 = AB^2 + AE^2$$

(c) To construct a square whose area is equal to the difference of two squares.

In the above figure, cut off $FG=FE$. Then the square on DG is the required square (Bodhayana, Apasthamba, Kathyayana) for,

$$DG^2=FG^2-DF^2=AB^2-AE^2$$

(d) To construct a square equal to a given rectangle.

From the longer side AB of the rectangle $ABCD$, cut off $AE=AB$, and complete the square $AEFB$. Bisect the rectangle $CDEF$ by drawing GH parallel to AB. Complete the square $AGKL$. Then, rect. $ABCD$ $=$sq. $AGKL-$sq. $HKMF$. Apply now the construction of (c), i.e. cut off $LP=LK$, and draw PQ parallel to AB. Then the square on LQ is the required square.

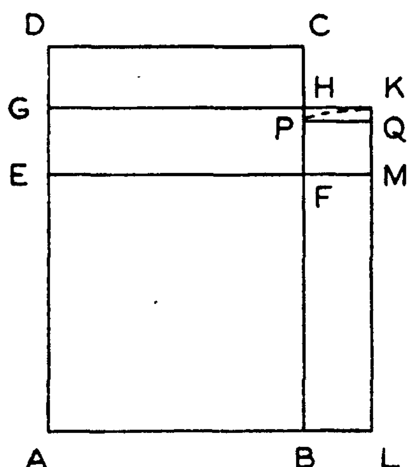


FIG. 4.

This construction is evidently more complicated than Euclid's method, which is simple and elegant.

There are many other constructions : to construct a triangle equal to a rectangle, to construct a square equal to a rhombus, to construct a square equal to the sum of two triangles or two pentagons, etc.

13. The essentially arithmetical background of the Sulva mathematics must be contrasted with the essentially geometrical background characteristic of Greek mathematics. Simple fractions and operations on them are available in the Sulvas. We meet with fractions like $\frac{3}{8}$ (Thri Ashtama, त्रि अष्टम), $\frac{2}{7}$ (Dwi Saptama, द्वि सप्तम), $\frac{3}{4}$ (Chaturbhagona, चतुर्भागोन). These are not unit fractions only, as were used in ancient Egypt, Babylonia and China. Apasthamba gives the area of a square of side $1\frac{1}{2}$ purushas as $2\frac{1}{4}$, and that of a

14. The above facts make it clear that the Indians were the first nation to use irrational numbers. The Greeks also used irrational numbers. If AB is a given segment, Pythagoras and others described the methods of constructing segments of length $\sqrt{2}\ AB$, $\sqrt{3}\ AB$, $\sqrt{5}\ AB$, etc. But no rational approximations to $\sqrt{2}$, $\sqrt{3}$ etc., are found in Greek mathematics, nor are there any problems involving arithmetical operations on irrational numbers. This is easily explained, because the requisite knowledge of arithmetic was not available to the Greeks. It will also be borne in mind that according to unprejudiced estimates, the Sulva Sutras are about two or three centuries prior to Pythagoras.

A strange argument is advanced for trying to deprive the credit of this achievement to the Hindus. It is argued that the Vedic Hindus believed that $\sqrt{2}$ was exactly equal to $1+\frac{1}{3}+\frac{1}{3\cdot4}-\frac{1}{3\cdot4\cdot34}$, and hence the conception of an irrational number did not come in. It is unnecessary to refute this argument at length. Three different approximations to $\sqrt{2}$ are mentioned above, and these are not equal to one another. The people who knew so much mathematics, and who wrote $\sqrt{7\frac{1}{9}}=2\frac{2}{3}$ would certainly not take $\sqrt{61}=7\frac{5}{6}$ (*Mānava Sulva*) as exact. The very fact that they realised that $\sqrt{32}$, $\sqrt{61}$ etc. could not be exactly determined amounts to the introduction of irrational numbers. Formal concepts, by way of a mathematical definition and a theory based on this, are of course of recent origin, as propounded by Dedekind, Cantor and Weierstrass. But *the credit of using irrational numbers for the first time must go to the Indians.*

15. The problem of constructing a circle equal in area to a given square, and of constructing a square equal in area to a given circle are problems which have come down from the times of the *Samhitas*. Since the area of a circle of radius r is πr^2, the problem is essentially equivalent to the determination of π. But the concept of π, in other words the knowledge that the ratio of the circumference of a circle to its diameter is constant is not explicitly to be found in the Sulvas, though the notion is evident from consideration of similar figures. Bodhayana gives the following constructions to the above problems.

(i) From the centre O of the square, draw a circle of radius

OC to cut the line through O parallel to AB in E. Cut off $LM=\frac{1}{3}LE$, where L is the point on BC. Then the circle with OM as radius is equal to the square $ABCD$.

If $AB=2a$, we get $OM=\frac{a}{3}(2+\sqrt{2})$. This gives the value for π, viz. $\pi=18(3-2\sqrt{2})$.

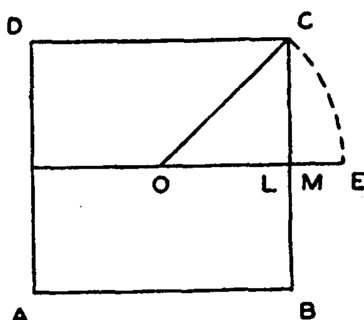


FIG. 5.

(ii) Divide the diameter of a circle into 8 parts, and one of these parts again into 29 parts, and reject 28 out of these, and a further $\frac{1}{6}-\frac{1}{6\cdot8}$ of it. We obtain in this way the side of the equivalent square. Expressed in symbols, if d is the diameter of the circle, the side of the square is given by

$$2a=\frac{7}{8}d+\left[\frac{d}{8}-\left\{\frac{28}{8\cdot29}d+\frac{d}{8\cdot29}\left(\frac{1}{6}-\frac{1}{6\cdot8}\right)\right\}\right].$$

Both these constructions give the value $\pi=3{\cdot}088$ roughly. The approximation is not good, though it would serve the purpose for which it was intended, viz. constructing a circular altar of the same area as that of a square. The method of deriving the above formulæ is one of conjecture, but will have little interest on account of the crudeness of the approximation.

16. We now consider some brick constructions, detailed in the Sulvas, which call forth a certain amount of skill in mathematical computation, and provide interesting examples of simultaneous indeterminate equations.

The altar pertaining to Gārhapatyāgni is to be constructed so as to have five layers of bricks one over the other, so as to contain 21 bricks in each layer, and so as to provide an area of one square Vyāyām. Any one who has seen constructions in brick knows that when one layer is built upon another, the cleavage between two bricks of the bottom layer is to be covered by the solid material of the superimposed brick. Bearing this engineering principle in mind, the construction is to be carried out. The bricks may be

rectangular or square. and may be constructed so as to have any required size. The altar itself may be circular or square.

Let us have in any one layer, x square bricks of one size, and y square bricks of another size. Let their lengths be $\frac{1}{m}$ and $\frac{1}{n}$ vyayam respectively. Then the above description leads to the simultaneous equations

$$x+y=21\ ;\ \frac{x}{m^2}+\frac{y}{n^2}=1.$$

Solving,

$$x=\frac{m^2(21-n^2)}{m^2-n^2},\quad y=\frac{n^2(m^2-21)}{m^2-n^2}.$$

If we assume $m>n$, we must therefore have $m^2>21>n^2$. This gives $n\leqslant 4$. Substituting $n=1, 2, 3, 4$ in turn, we can find by trial the value of m which makes the values of x and y integral. We thus get the following solutions :

$$m=6,\ n=3,\qquad x=16,\ y=5$$
$$m=6,\ n=4,\qquad x=9,\ y=12.$$

This explains the construction formulated for the altar by Bodhayana. Construct three types of bricks of square size, with lengths $\frac{1}{6}$th, $\frac{1}{4}$th and $\frac{1}{3}$rd of a Vyāyām. Then the first layer is to be constructed out of 9 bricks of the first type, and 12 of the second. The second layer should contain 16 bricks of the first type and 5 of the third. Then the third and fifth layers are to be similar to the first, and the fourth similar to the second.

A more difficult mathematical problem is presented in the construction of the altar for the *Garuda-Chayana* (गरुड चयन). As before there should be five layers of bricks. Each layer is to contain 200 bricks, covering an area of $7\frac{1}{2}$ sq. purushas. Bodhayana constructs this using square bricks of four different sizes. Let a brick of each one of these sizes have the area $\frac{1}{m}$, $\frac{1}{n}$, $\frac{1}{p}$, $\frac{1}{q}$ respectively. If their numbers in any layer are x, y, z, w, we have then

$$x+y+z+w=200$$
$$\frac{x}{m}+\frac{y}{n}+\frac{z}{p}+\frac{w}{q}=7\tfrac{1}{2}$$

Bodhayana's solutions to these equations are

$$m=16,\ n=25,\ p=36,\ q=100\ ;\qquad \left.\begin{array}{c} x=24,\ y=120,\ z=36,\ w=20 \\ \text{or} \\ x=12,\ y=125,\ z=63,\ w=0 \end{array}\right\}$$

In other words, Bodhayana uses square bricks of lengths $\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{10}$ of a purusha, the numbers of bricks of these sizes being in either of the alternate ways mentioned. Bodhayana also explains that it is possible to use rectangular bricks. These can be further split up into square bricks, but Bodhayana does not advocate this, as this will involve the construction of a large number of types of bricks. We thus have the solutions

$$m=25,\ n=50,\ p=\tfrac{50}{3},\ q=100\ ;\qquad \left.\begin{array}{c} x=160,\ y=30,\ z=8,\ w=2 \\ \text{or} \\ x=165,\ y=25,\ z=6,\ w=4 \end{array}\right\}$$

$$\left[\tfrac{1}{50}=\tfrac{1}{100}+\tfrac{1}{100}\ ;\ \tfrac{3}{50}=\tfrac{1}{100}+\tfrac{1}{25}+\tfrac{1}{4}\right]$$

Apastamba (श्रापस्तंब) uses five types of bricks in the construction of the above altar. This requires the solution of the simultaneous equations

$$x+y+z+w+u=200,\ \frac{x}{m}+\frac{y}{n}+\frac{z}{p}+\frac{w}{q}+\frac{u}{r}=7\tfrac{1}{2}. \qquad \cdots \quad (1)$$

Apastamba does not explain his method of construction in clear terms, and commentators have given several solutions to these equations. The dates of these commentators are not known, but presumably they are very much later than the period of the Sulva Sutras.

Name of commentator	m	n	p	q	r	x	y	z	w	u
Karavindaswamy	16	25	64	100	144	67	58	48	18	9
,,	16	25	36	64	100	12	157	9	0	22
Kapardiswami	,,	,,	,,	,,	,,	10	159	9	8	14
Sundara raja	,,	,,	,,	,,	,,	70	45	9	56	20
,,	16	25	64	100	144	74	45	52	20	9
,,	,,	,,	,,	,,	,,	77	42	40	32	9

These do not exhaust the solutions to the above equations. The method of solving such equations has been explained by K. N. Kamalamma*, and will be outlined below.

* *Bull. Calcutta Math. Soc.*, 40 (1948), 140—44.

Let r be the smallest of the square numbers m, n, p, q, r. Multiplying the second equation in (1) by r, and subtracting from the first, we easily see that

$$7\tfrac{1}{2}r-200<0.$$

Hence the possible values of r are 25, 16, 9, 4, 1. Similarly the largest of these squares must be $>200/7\tfrac{1}{2}$ i.e. $\geqslant 36$. Let us try the values

$$m, n, p, q, r=1, 25, 4, 9, 36.$$

Put $\frac{x}{m}=k_1$, $\frac{y}{m}=k_2$. We get

$$z+w+u=200-k_1-25k_2$$

and $$9z+4w+u=270-36(k_1+k_2)$$

$$\therefore\quad 8z+3w=70-35k_1-11k_2.$$

k_1+25k_2 and $36(k_1+k_2)$ are necessarily integers. k_1 itself is an integer, since $m=1$. It follows that k_2 is also an integer.

Taking $k_1=1$, we can have $k_2=1$, 2 or 3, but $k_2=2$ or 3 leads to $8z+3w=13$, or $8z+3w=2$, which have no solutions. Hence $k_1=k_2=1$, and we thus obtain the solution

$$x, y, z, w, u=1, 25, 0, 8, 166$$

$$m, n, p, q, r=1, 25, 4, 9, 36.$$

Similarly, when m, n, p, q, r are taken as 25, 4, 16, 9, 36, we put $\frac{x}{25}=k_1=$ an integer, and $\frac{w}{9}+\frac{u}{36}=k_2$

$$\therefore\quad 4y+z=120-16(k_1+k_2).$$

Giving different values to k_1, k_2 such that $16(k_1+k_2)$ in an integer $\leqslant 120$, we can find the values of y, z and w, u.

Similar empirical methods can be used to construct a good many solutions to the problem. There is no doubt that some such empirical method should have been employed by the ancient sages for the solution of the problem. Kamalamma, however, proves that the total number of solutions is finite.

The Sulva Sutras contain several other constructions involving simultaneous equations of similar type.

CHAPTER III

THE MATHEMATICS OF THE JAINAS

17. If mathematics played an important part in the religious observances of the ancient Aryans or Hindus, on account of the precision it gave in the construction of their sacrificial altars, besides providing the means for an accurate determination and measurement of time, equally did mathematics play an important role in the Jaina religion. The Jainas went to the extent of regarding mathematics as an integral part of their religion. A section of their religious literature was named *Ganitānuyoga* (literally, the system of calculation). Mahavira, the founder of the Jain religion was well versed in mathematics.

The original mathematical works of the Jainas have not come to light, and a considerable amount of search and research about the ancient works of the Jainas is necessary. Our present knowledge about them is almost entirely based on the available commentaries.*

Amongst the religious works of the Jainas, those that are important from the view-point of mathematics are *Sūrya Prajnapti*, *Jamboo Dwīpa Prajnapti*, *Sthānānga Sūtra*, *Uttarādhyayana Sūtra*, *Bhagawatī Sūtra*, and *Anuyoga Dwāra Sūtra*. The approximate date of the first two works is about 500 B. C., and the rest may be of about 300 B. C.

One of the great preceptors of the Jainas, by name Bhadrabāhu came down from Bihar (Magadha), and settled down at Sravana Belagola in the Mysore State, in about 313 B. C. His commentary on the Sūrya Prajnapti, and a work on astronomy by name *Bhadra bāhavī Samhita* have yet to be unearthed.

A work of considerable importance is the *Tattwārthādhigama-Sūtra Bhāshya* of Umāswāti. There is some doubt about his date. According to Swetāmbar Jains (one of the two main sects amongst the Jains), Umāswāti was born at a place called Nyagrōdhika, and

* See B. Datta : *Bull. Calcutta Math. Soc.* xxi, 115-45.

lived about 150 B. C. The name is a combination of the mother's name Umā and the father's name Swāti. It is definite that he lived in the city of Kusumapura, which later became to be called Pātaliputra (near Patna). The existence of a school of mathematics at Kusumapura or Pātaliputra from about the first century B. C. seems to be fairly certain, and the school continued there for several centuries, for Aryabhata (born 476 A. D.) also belongs to the Kusumapura school.

Umāswāti's name has come down to us as a great writer on Jaina metaphysics, but he is not known to be a mathematician. It is to be concluded that the mathematical formula and results quoted in his work were taken from some treatise on mathematics extant in his time.

18. Amongst the mathematical results contained in the *Tattwārthādhigama Sūtra Bhāshya* are a number of mensuration formulae, which include the following :

(1) circumference of a circle $=\sqrt{10}\times$ diameter

(2) area of a circle $=\frac{1}{4}$ circumference $\times$ diameter

(3) chord $=\sqrt{4 \text{ shara } (\text{diameter} - \text{shara})}$

(4) shara $=\frac{1}{2}[\text{diameter} - \sqrt{(\text{diameter})^2 - (\text{chord})^2}]$

(5) arc of segment (less than a semicircle)

$$=\sqrt{6(\text{shara})^2 + (\text{chord})^2}.$$

(6) diameter $=\dfrac{(\text{shara})^2 + \frac{1}{4}(\text{chord})^2}{\text{shara}}$.

Shara (शर) means the height or the arrow of the segment.

All these formulae are restated in another work of Umāswāti, the *Jambūdwīpasamāsa*.

The first of the above-mentioned formulae gives $\sqrt{10}$ as the value of π, an approximation which the Jainas have used from 500 B. C. till the 15th century A. D. The *Sūrya Prajnapti* gives two values for π, viz. $\pi=3$, and $\pi=\sqrt{10}$. The former is due to earlier writers and has been discarded by the author. These formulae come into being in connection with the Jaina cosmography system, according to which the Jambūdwīpa (the earth) is a circle of diameter 100,000 yōjanas, and is divided into seven parts by six mountain ranges running parallel, from east to west, at regular

intervals. The circumference of the earth is stated in the *Jambūdwīpa Prajnapti* (about 500 B. C.) and other works, as 316227 yōjanas 3 gavyūtis 128 dhanus and a little over $13\frac{1}{2}$ angulas, while the area is 7905694150 yōjanas, 1 gavyūti, 1515 dhanus, and 60 angulas (roughly). According to the square root method, the value of $\sqrt{10}$ is 3.1622776601683...... . This clarifies their table of measurements : 1 yōjana = 4 gavyūtis, 1 gavyūti = 2000 dhanus, 1 dhanu = 100 angula*. If the square root value of $\sqrt{10}$ is used, the circumference is slightly less than......$13\frac{1}{2}$ angulas, instead of being slightly greater. It indicates that the value of $\sqrt{10}$ was obtained correct to nearly 13 places of decimals. Such a close determination of the value of $\sqrt{10}$ in 500 B. C. is commendable. The irrationality of $\sqrt{10}$ also seems to have been clearly understood.

As stated above, the earth was divided into seven parts, and the southernmost segment is the *Bharatavarsha* (India). According to the *Jambūdwīpa-Prajnapti*, we have the following dimensions for this segment :

The shara or height or breadth = $526\frac{6}{19}$ yōjanas. The length i.e. the chord is a little over $14471\frac{6}{19}$ yōjanas. The length of the southern boundary i.e. the arc = $14528\frac{11}{19}$ yōjanas. We may verify one of the formulæ here, applying the mensuration rules quoted earlier. Thus,

$$\text{chord} = \sqrt{4\ \text{shara}\ (\text{diameter} - \text{shara})}$$

$$= \sqrt{\left\{4 \times \frac{10000}{19} \times \frac{1890000}{19}\right\}}$$

$$= \frac{20000}{19}\sqrt{189} = \frac{20000}{19} \times 13{\cdot}74772...$$

$$= \frac{274955}{19}\ (\text{approx.}) = 14471\frac{6}{19}.$$

It is fairly certain that the methods of finding the square root of any number to any degree of approximation were known, though decimals were not in vogue.

* The angula must be synonymous with its present usage, the inch. The dhanu may be practically identical with the Vedic unit of length, the Purusha or Vyayama.

Formula (4) has evidently been obtained from (3) which can be written $c=\sqrt{4h(d-h)}$, where c, h, d stand for the chord, the height and the diameter. Formula (4) follows from the solution of the quadratic equation $c^2=4hd-4h^2$, giving $h=\frac{1}{2}(d\pm\sqrt{d^2-c^2})$. That the Jainas knew how to solve quadratic equations is clear. This does not mean much, since examples of solutions of quadratic equations in China and in Babylonia, by about 2000 B. C., or even earlier are available. One or two trivial examples will be found in the Sulva Sutras, but no specific non-trivial example is available in India, prior to Jaina Mathematics.

The formula $c=\sqrt{4h(d-h)}$ also points to a knowledge of the geometrical properties of circles, leading up to the theorem, the square on chord = the rectangle contained by the segments of the diameter perpendicular to the chord.

19. The above heavy calculations and close approximations are enough circumstantial evidence of a knowledge of the decimal place-value system of numeration, and the arithmetic developed according to it. The actual numerals employed are not known, but the Jaina works refer to a very large number of names giving the positions (Sthāna, or place) in the numeral system. We have Eka (ऽक), Dasa (दश), Shata (शत), Sahasra (सहस्र), Dasa-Sahasra (दशसहस्र), Dasa Shata Sahasra (दश शतसहस्र), Kōti (कोटि), Dasa Kōti (दशकोटि), Shata Kōti (शतकोटि), which are respectively equivalent to one, ten, hundred, thousand, ten thousand, ten hundred thousand, 10^7, 10^8 and 10^9. The deviations from the Vedic terminology (ch. I, §3) will be noted. The combination terms like Dasa Sahasra, Dasa Kōti, etc. in place of the distinct terms for them used in the Vedas are an indication that large numbers were of frequent usage, and hence the combination terms were preferred to the distinct names given in the Vedas.

The Jainas required very large numbers for their measurements of space and time. No nation has used such large numbers as the Jainas and the Buddhists. The Buddhistic work *Lalitha Visthara* of the first century, B. C., gives the names of numbers up to 10^{53}. In Kachchāyana's Pali grammar, the number 10^{140} is called Asankhyēya (lit. uncountable, असंख्येय). According to a certain

measurement of time,

one pūrvi (पूर्वि) = 75600,000 000,000 years*

one Shirsha Prahelika (शीर्ष प्रेहेलिका)

$= (8,400,000)^{28}$ pūrvis.

This number contains 194 digits.

20. The introduction of such large numbers led the Jains to a conception of infinity, which if not mathematically precise, is by no means crude. Numbers were classed as enumerable (संख्येय), unenumerable (असंख्येय) and infinite (अनंत). Enumerables were obtained by writing down the numbers commencing from 2 (1 was ignored), till the highest numeral was reached. How to reach this? "Consider a trough whose diameter is of the size of the earth (100,000 yojanas). Fill it up with white mustard seeds counting them one after another. Similarly fill up with mustard seeds other troughs of the sizes of the various lands and seas. Still it is difficult to reach the highest enumerable number." After "attaining" this number, call it N, a sequence of operations is explained in order to reach infinity, which may be exhibited in the following form:

$$2, 3, \ldots\ldots, N$$
$$N+1, N+2, \ldots\ldots, (N+1)^2-1$$
$$(N+1)^2, \ldots\ldots, (N+1)^4-1$$
$$(N+1)^4, \ldots\ldots, (N+1)^8-1$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Infinity itself was of five kinds, infinite in one direction, infinite in two directions, infinite in area, infinite everywhere, infinite perpetually (एकतो अनंतं, द्विधानंतं, देशविस्तारानंतं, सर्व-विस्तारानंतं, शाश्वतानंतं). Thus they combined the idea of infinity with that of dimension defining infinity in one, two, three and infinite dimensions.

21. *Laws of indices.* Without a convenient notation for indices a comprehensive formulation of the laws of indices is not to be expected. But noteworthy steps in this direction made by the Jains will be found interesting. The *Anuyōgadwāra Sūtra*, about the first

* H. R. Kapadia's Introduction to his *Ganita tilaka* of Sripati. Gaekwad's Oriental Series, No. 78.

century B. C., enumerates the powers and the roots of numbers, as the first square, second square, third square, ..., first square root, second square root etc. Performed on the number a, this means $(a)^2$, $(a^2)^2$, $(a^4)^2$, ..., and $\sqrt{a}$, $\sqrt{(\sqrt{a})}$, $\sqrt{\{\sqrt{(\sqrt{a})}\}}$, etc., i.e. the numbers a^2, a^4, a^8, ..., and $a^{\frac{1}{2}}, a^{\frac{1}{4}}, a^{\frac{1}{8}}$, etc. While this gives only in positive and negative powers of 2, another work, the *Uttarādhyayana Sūtra* (300 B. C., or earlier, and thus much earlier to the above work) gives other powers : thus varga=square, ghana (घन)= cube, varga-varga=4th power, ghana-varga=6th power, ghana-varga-varga=12th power, third varga-mūla-ghana$=\{(a^{\frac{1}{8}})^3\}^3=a^{\frac{3}{8}}$. In the *Anuyōgadwāra Sūtra*, we meet with statements such as, "the first square root multiplied by the second square root, or the cube of the second square root ; the second square root multiplied by the third square root, or the cube of the third square root." Expressed in symbols, these mean

$$a^{\frac{1}{2}} \times a^{\frac{1}{4}} = \left(a^{\frac{1}{4}}\right)^3 \; ; \; a^{\frac{1}{4}} \times a^{\frac{1}{8}} = \left(a^{\frac{1}{8}}\right)^3.$$

The *Anuyōgadwāra Sūtra* gives the total population of the world as follows : "it is a number which in terms of the denominations Kōti-Kōti etc., has 29 places. It is a number obtained by multiplying the sixth square by the fifth square, or a number which can be divided (by two) 96 times." This means that the population is $2^{64} \times 2^{32} = 2^{96}$, which can be verified to have 29 digits.

There is therefore no doubt that the place-value system of numeration was widely in use by about the first century B. C., and that the Jainas knew the laws of indices at least in simple cases.

22. According to the *Sthānānga Sūtra* (about 300 B. C.), the topics for discussion in mathematics (Samkhyāna) are ten in number : *Parikarma, vyavahāra, rajju, rāsi, kalāsavarma, yāvat-tāwat, varga, ghana, varga-varga* and *vikalpa.* The exact meanings of some of these terms are not clear, and have been the subject of controversy. Further research on ancient Jaina mathematics is necessary to clarify the matter. But in the light of the mathematics that is at present available from the works mentioned previously, and in the light of the usage of some of these words in later Hindu mathematical works, we could reasonably interpret these terms as follows : *Parikarma*

refers to the four fundamental operations of arithmetic, viz. addition, subtraction, multiplication and division. *Vyavahāra* means the applications of arithmetic to concrete problems (applied arithmetic). *Kalāsa varma* refers to fractions. These three words have been used in exactly this sense in the *Ganita-Sāra Sangraha*, a work of a later Jaina mathematician Mahāvīra (850 A. D.). *Rajju* is the ancient Hindu name for geometry, which was called the *Sulva* in the Vedic period. *Rāsi* means heap and may refer to measurements of grain etc., or it may refer to mensuration of plane and solid figures. The terms *varga*, *ghana*, *varga-varga* need not be translated literally as square, cube, square-square. The first two may be abbreviations not only for the square and the cube, but also the square root and the cube-root. In the light of what has been said in §21, *varga-varga* may be a general name to connote the subject of getting higher powers and roots, in brief the subject-matter of indices. The word *yāvat-tāwat* is the word for the unknown quantity in ancient Hindu mathematics, and provides the algebraic symbol yā (या). It is difficult to account for this topic except by saying that it means the science of algebra, in however rudimentary form it may have existed. Besides the problems on indices in a general form, this subject may have included solutions of problems of arithmetric by assuming unknown quantities, and simple summations.

We now come to the very important topic *vikalpa*, which is the Jaina name for the subject of permutations and combinations. This subject has been treated in some detail for the first time in the ancient Jaina works, and general formulæ were given later by Mahavira. Simple problems are set forth in the *Bhagabati Sutra* (300 B. C.), such as the number of philosophical doctrines that can be formulated by combining a certain number of basic doctrines, taking one, two, three or more at a time. Similarly we have calculations of the groups that can be formed out of the five senses, selections that can be made out of a given number of men and women, etc. The corresponding formulæ have been correctly given, which in symbols are

$${}_nC_1=n,\ {}_nC_2=\frac{n(n-1)}{1.2},\ {}_nC_3=\frac{n(n-1)(n-2)}{1.2.3},$$

$${}_nP_1=n,\ {}_nP_2=n(n-1),\ {}_nP_3=n(n-1)(n-2).$$

These have been given for $n=2, 3, 4$, and the author observes that "in this way, 5, 7, ..., 10, etc., enumerable, unenumerable or infinite number of things may be mentioned. Taking one at a time, two at a time, ... ten at a time, twelve at a time, as the number of combinations are formed, all of them must be worked out." Apart from the generalisation attempted, the idea of applying the principle to different kinds of infinities, or different dimensions is noteworthy. A study of *Anuyōga dwāra Sūtra* and its commentary by Hemachandra Sūri (b. 1089 A. D.) makes it clear that the general value of ${}_rP_r$ viz. $r!$ was known, as also its proof. They could also write down the number of permutations with a given element occupying a given place, or with two given elements occupying given places, and so on.

23. The notions of permutations and combinations aretr aceable in India even before the advent of Jainism. But credit is due to the Jains for having treated the subject as a topic in mathematics, and for having worked out the general formulæ, by the time of Māhavīra. In the Vedic period, one finds the idea about the number of ways in which the metre or *Chchandas* (छंदस) can be varied. In Sushruta's medicinal work*, written about the 6th century B. C., it is said that 63 combinations may be made out of the six different *rasas* (रस) or tastes (bitter, sour, saltish, astringent, sweet, hot) by taking the *rasas* one at a time, two at a time, etc. For we obtain thus 6, 15, 20, 15, 6 and 1 combinations, which total up to 63. The word *Vikalpa* for combinations is therefore traceable prior to Jainism.

A writer on prosody, by name Pingala, in the third century B. C. considers in his *Chchandas-Sūtra* (छंदस्सूत्र) the method of finding the number of combinations obtainable by taking one letter, two letters etc. (ēkaka samyoga, dwika samyoga† etc.), out of a given number of letters. The meaning of the rule is difficult to understand. A commentator of the 10th century A. D., by name Halāyudha explains the meaning as follows: First draw a square (Fig. 6). Below it, and starting from the middle of the lower side, draw two

* *Sushruta Samhita*, ch. 63, *Rasabheda Vikalpadhyaya*.

† The terminology is identical with that in *Bhagawati Sutra*, referred to earlier, which is also of about the same date.

squares. Similarly, draw three squares below these, and so on, as in the figure. Write the number 1 in the middle of the top square, and inside the first and last squares of each row. Inside every other square, the number to be written is the sum of the numbers in the two squares above it and overlapping it. The diagram so formed is called *Mēru Prastara* (मेरुप्रस्तार). The numbers in the *r*th row are ${}_rC_1$, ${}_rC_2$, ${}_rC_3$, ..., ${}_rC_r$ in order. The diagram clearly illustrates the formula ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$, and may have been constructed on its basis.

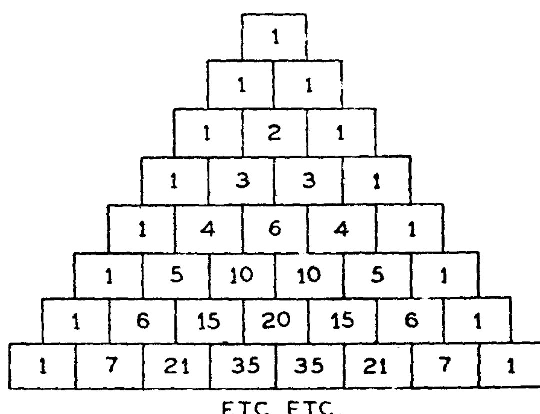


FIG. 6.

The diagram is to be compared with Pascal's triangle, but the *Mēru-Prastara* rule is simpler than that of Pascal. The *Mēru-Prastara* rule is based on the simple formula ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$, while the rows in Pascal's triangle are the sums of one, two, three etc. elements of the previous row, and the numbers ${}_nC_1$, ${}_nC_2$, ... are arranged along diagonals. It should be noted that the *Mēru-Prastara* rule was formulated many centuries (even the commentator Halāyudha is 6 centuries prior to Pascal) before Pascal.

CHAPTER IV

THE BAKHSHĀLI MANUSCRIPT

24. In the year 1881 A. D., a farmer in the course of excavation found a manuscript of a work on mathematics, written on birch-bark, at a village called Bakhshāli near Peshawar. Only about 70 leaves, some of which were mere scraps, were available, the greater portion of the manuscript having been lost. The full text has not been found. It has been written in the Shārada script in the Gātha language (a modified form of Prākrit). The work has been printed and published by the Government of India with photographic facsimiles and transliteration of the text, together with an elaborate introduction by G. R. Kaye. But unlike Colebrooke, Maxmüller and many other Western scholars whose painstaking work in bringing out works on ancient Indian culture in a truly scientific spirit has earned for them the lasting gratitude of the Indians, G. R. Kaye is looked upon as one whose commentaries and interpretations on Indian mathematics are generally prejudiced and distorted.

The *Bakhshāli* manuscript is a compendium of rules and illustrative examples together with their solutions. It is devoted mostly to arithmetic and algebra, with just a few problems on geometry and mensuration. It is likely that the parts dealing with these topics have been lost. The topics on arithmetic include fractions, square root, profit and loss, interest and rule of three. The topics on algebra include simple and simultaneous equations, quadratic equations, arithmetic and geometric progressions. The sections dealing with the several topics are not well-defined and problems are often mixed up.

25. Nothing is known about the date of the work or about the author. Western scholars estimate the date by about the third or fourth century A. D., though G. R. Kaye would like to assign it to the twelfth century, basing his arguments on the "script, the language and the contents of the work." Bibhutibhusan Datta

argues* that in the absence of direct evidence, the age should be estimated on historical grounds and not on literary and palaeographic evidence. The mathematical principles, symbols and terminologies used in the work are the best guides, according to Datta, who therefore agrees that the date is about the third or fourth century A. D. Evidence based on language is of course important, but this should be given by experts in language, and may not be always conclusive.

But the date of the manuscript becomes unimportant when we recognize that the work which has now been found is only a copy or perhaps a commentary (करणग्रंथ) of an older work. The manner of its composition and the very elaborate details of explanation, repetitions, and cross-references to earlier pages are all circumstantial evidence to come to the conclusion that the whole work is that of a commentator. Further than this, the manuscript exhibits the writings of more than one scribe, possibly as many as five. In one place, there is an observation in between two lines, that a certain rule is wrong. No author would pass over a mistake in his work, saying that it is wrong. The observation can only be that of the scribe who copied the work. There is also a statement that the "work was written (लिखीतम्) by a Bhāmhana mathematician, son of Chajaka, for the education of the son of Vasishta and for the benefit of succeeding generations." If Chajaka's son had been the author himself, he would have used the words *kritam* or *virachitam* (कृतं, विरचितं) (composed) instead of *likhitam*.

Circumstantial evidence is quite strong to indicate that the Bakhshāli Mss. is a commentary of an earlier work. The original work itself was not a systematic treatise, and rules relating to the same topic were not collected together. The commentator or commentators have tried to improve the order, without disturbing the original too much. Hence they have commented on a number of sūtras together, some of which occur later in the work.

The date of the mathematics contained in the Bakhshāli manuscript is therefore far more important than the date of the Ms. itself, but a precise estimate of the former date may be possible only when further such manuscripts come to light.

A characteristic feature of the Bakhshāli work is its elaborate

* B. Datta : *Bull. Calcutta Math. Soc.*, xxi, 1-60.

exposition, which as we have already remarked may be due to its having been a commentary. There are a few technical words which are special to this work, which have disappeared in later Hindu writings. Thus the reduction of fractions to a common denominator is *Savarnana* (सवर्णन i.e. making them of the same class), but in the Bakhshāli work, it is *sadrisī-karana*, or *hara-sāmya-karana* (सदृशीकरण, making them similar ; हरसाम्यकरण, making the denominators equal). In later works, the statement of a problem is *nyāsa* (न्यास), while in the Bakhshāli work, it is more frequently called *sthāpana* (स्थापन), and occassionally as nyāsa, or *nyāsa-sthāpana*. The usual Hindu term for series is *srēdhi* (श्रेढ़ि), but we find in this work the terms *varga* (वर्ग, group), or *partha* (पार्थ, derived from prtha, पृथ=several), as also *rūpana-karana* (रूपणकरण, summation). The terminology in the Bakhshāli work was evidently in a process of evolution, till it reached a generally accepted form by the time of Aryabhata. The subject-matter of *kuttaka* (indeterminate equations of the first degree) is found to have been discussed for the first time by Aryabhata, and is found in the mathematical works of all later writers. Its complete absence in the Bakhshāli Ms. is also a pointer that the work was prior to Aryabhata, though we do not know what subjects had been discussed in the lost pages of the work:

26. We now consider the mathematical contents of the work. The reduction of a number of fractions to a common denominator, and hence their summation was known to the Bakhshāli author. The following examples are found in the work :

(1) To find the sum of $\frac{2}{1}$, $1\frac{1}{2}$, $1\frac{1}{3}$, $1\frac{1}{4}$, $1\frac{1}{5}$.

They are first reduced to a common denominator (*sadrisam kriyate*), so as to become

$$\frac{120}{60},\ \frac{90}{60},\ \frac{80}{60},\ \frac{75}{60},\ \frac{72}{60}$$

respectively. The sum is stated to be $\frac{437}{60}$.

(2) The sum of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{4}$, $\frac{3}{5}$ is similarly obtained as $\frac{163}{60}$.

(3) The expression* $\frac{13\frac{1}{3}}{3\frac{1}{8}}+\frac{13-\frac{1}{4}}{8\frac{1}{2}}+\frac{1\frac{1}{3}}{3\frac{1}{5}}+\frac{\frac{1}{2}}{1\frac{1}{2}}+\frac{1}{5\frac{1}{3}}+\frac{2\frac{1}{4}}{5}+\frac{12\frac{1}{2}}{33\frac{1}{3}}$

is equal to $\frac{1807}{240}$.

* The manuscript is erroneous. The expression is corrected by Datta, and tallies with the answer.

The notation for a negative sign used in the Bakhshāli manuscript is to use a + sign (cross) after the concerned number. Thus 11 7+ means 11 − 7. This is very different from the sign in later Hindu mathematics, which is to use a dot over the number, thus 11 $\dot{7}$. All other arithmetical operations are generally indicated either by writing down the words in full, or by their first syllables or where the context is clear, by simply writing down the numbers one below the other. The origin of the symbol + for subtraction may be through the word kshaya (क्षय), since kṣa in the Brahmi characters or in the Bakhshāli characters differs from the symbol + in only having a little flourish at the lower end of the vertical line.

27. *The symbol for the unknown quantity.* The well-known symbol for this in Hindu mathematics is *yāvat-tāwat* abbreviated into *yā*. *Yāvat-tāwat* literally means "what is, that", i.e. "that which", or "that which is required". In ancient Jaina mathematics, *yāvat-tāwat* means the science of algebra. When exactly the symbol *yā*, or for the matter of that, any definite symbol was used for the unknown quantity is not clear. Amara Sinha (about 400 A. D.) explains in his lexicography, Amara Kōsa (अमर कोश) that *yāvat-tāwat* means measure or quantity (यावत्तावच्चसाकल्येऽवधौ मानेऽवधारणे). Amara Sinha is a Jain, and he must surely have been acquainted with the Jaina works on mathematics. The word *yadṛccha* (यदृच्छा) of synonymous import is used in the Bakhshāli manuscript, in place of *yāvat-tāvat*, and the symbol used for it is to put a zero (0) in its place. *Yadṛcchā vinyase sūnye* (यदृच्छा विन्यसे शून्ये), i.e. put a zero in the place for the unknown or desired quantity. The symbol 0 has of course also been used to denote zero, when it occurs. This symbol for the unknown is seen to be lurking even in later writers like Sridhara (750 A. D.) and Bhāskara (1150 A. D.), who use well-defined notations for the unknown quantity in their algebras but use o for the unknown in their arithmetical works, where algebraic symbols are not permitted. Thus, in Sridhara's *Trisatika* we have

ādi 20|*u* o |gaccha 7|ganitam 245

which means, the first term of an arithmetic progression is 20, the common difference is unknown, the number of terms is 7, the sum is 245.

We conclude from all these facts as follows :

(1) the word *yāvat-tāvat* for the unknown quantity and for algebra was in existence from the Jaina period, but the author of the Bakhshāli work was not aware of the Jaina works.

(2) the symbol *yā* for the unknown is of very much later origin, later than Brahmagupta (628 A. D.).

The absence of proper symbolism for the unknown quantities leads to a certain amount of ambiguity in the representation of equations, and the interpretation has to be made suitably according to the context. Thus

⊃	5	*yu*	*mū*	○	*sa*	○	7+	*mū*	○
1	1			1		1	1		1

means the quadratic indeterminate equations

$$\sqrt{x+5}=s, \qquad \sqrt{x-7}=t.$$

$\overset{\circ}{1}$ has to be read as a "certain unknown", explaining the number 1 below the unknown, and distinguishing the symbol ○ from zero. In one instance, when there are a number of unknowns, the symbol ○ is dropped, and the first syllables of prathama, dvitīya, etc. (first, second, etc.) are employed.

9 pra	7 dvi	10 tr.	8 cha	11 pan	yutam jātam
7 dvi	10 tr.	8 cha	11 pan	9 pra	

pratyeka (kramēna) 16|17|18|19 (20).

This means,

9	of the	first	plus	7	of the	second	=16
7	,,	second	,,	10	,,	third	=17
10	,,	third	,,	8	,,	fourth	=18
8	,,	fourth	,,	11	,,	fifth	=19
11	,,	fifth	,,	9	,,	first	=20

In symbols, this would mean that the solutions of the equations $x_1+x_2=16$, $x_2+x_3=17$, $x_3+x_4=18$, $x_4+x_5=19$, $x_5+x_1=20$ are given by $x_1=9$, $x_2=7$, $x_3=10$, $x_4=8$, $x_5=11$.

The notations have not been consistent in other examples.

We may mention in this connection, a method called "regula falsi", or the rule of false position. This method is still used in schools

in certain problems on arithmetic, when the use of algebraic symbols has not been taught or is not permitted. As an example, out of a man's monthly income, one-half is spent on house-hold expenses, one-fifth on rent, one-fourth on other items, and there is a net saving of Rs. 20. What is his income ? We proceed, by "supposing" that his income is Rs. 100, calculating the saving, and applying rule of three finally. This method will be found in plenty in the writings of Mahāvira who applies them very cleverly even to geometrical problems, and of Bhāskara. When algebraic symbols and algebraic operations had not sufficiently developed, the method would be of great use, and so we find it extensively used in the Bakhshāli work.

27A. The formation of the terms of an arithmetic progression, and its sum to n terms is clearly given in the Bakhshāli work. There are also a few instances of series in G. P., and series formed by combining two series. We have remarked previously that the arithmetic and geometric progressions occur in Jaina mathematics.

28. Some indeterminate equations which are not quite elementary are solved in the Bakhshāli work.

To solve (in integers) $\sqrt{x+a}=s$, $\sqrt{x-b}=t$, the *Sūtra* (rule) given in the work giv s the solution

$$x=\left\{\tfrac{1}{2}\left(\frac{a+b}{m}-m\right)\right\}^2+b,$$

where m is an assumed i.e. arbitrary number. The method is as follows :

$$s^2-t^2=a+b. \quad \text{Put } s+t=\frac{a+b}{m},\ s-t=m.$$

Brahmagupta [*Brahma Sphuta Siddhanta*, xvii, 73, 84] gives exactly the same solution, but Mahavira's solution [*Ganita-Sāra-Sangraha*, vi, 275½] is clumsy and partial, viz.

$$x=\tfrac{1}{4}\left[\left\{\frac{(a+b)(1+\alpha)}{2}\right\}^2-\alpha\right]+1\pm\tfrac{1}{2}(a\sim b\mp\alpha),$$

where α is the excess of $a+b$ over the nearest even number, and where the upper or lower sign is to be taken according as $b>$ or $<a$.

The solution of the indeterminate equation

$$xy-bx-cy-d=0$$

is given in the Bakhshāli work as

$$x=\frac{bc+d}{m}+c,\ y=b+m,$$

where m is an assumed number. The method is as follows : We have $(x-c)(y-b)=bc+d$. Put $y-b=m$, therefore $x-c=\frac{bc+d}{m}$. Brahmagupta's solution (xvii, 63) is similar. Mahavira's solution is different and limited, and based on the regula falsi method.

29. *Approximate values of surds.* One of the most important mathematical contributions found in the Bakhshāli manuscript is a formula for the approximate evaluation of quadratic surds, a formula which is identical with what is known as Heron's formula. Heron was a Greek mathematician of about 200 A. D. The original Bakhshāli work may have been earlier to this. This approximation is further applied in this work to the calculation of certain errors.

The Bakhshāli *Sutra* translated, runs as follows :

"In the case of a non-square number, subtract the nearest square number ; divide the remainder by twice (the nearest square) ; half the square of this is divided by the sum of the approximate root and the fraction. This is subtracted, and will give the corrected root".

In symbols, we have

$$\sqrt{A}=\sqrt{a^2+r}=a+\frac{r}{2a}-\frac{\left(\frac{r}{2a}\right)^2}{2\left(a+\frac{r}{2a}\right)}.$$

We find the following applications of the rule in the course of the work.

$$\sqrt{41}=6+\frac{5}{12}-\frac{\left(\frac{5}{12}\right)^2}{2\left(6+\frac{5}{12}\right)}.$$

Similarly, the values of $\sqrt{105}$, $\sqrt{481}$, $\sqrt{889}$, $\sqrt{339{,}009}$.

Rodet holds* that a different approximation, but going up to

* L. Rodet : Sur une methode d'approximation des racines carrés conne da. l'Inde antèrieurement a la conquête d'Alexandre : *Bull. Soc. Math. d. France*, VII (1879), 98-102.

the fourth order, was known to the authors of the *Sulva-Sutras*. This is given by

$$\sqrt{a^2+r}=a+\frac{r}{2a+1}+\frac{\frac{r}{2a+1}\left(1-\frac{r}{2a+1}\right)}{2\left(a+\frac{r}{2a+1}\right)}+\epsilon,$$

where

$$\epsilon=\frac{r-\left\{\frac{r}{2a+1}+\frac{\frac{r}{2a+1}\left(1-\frac{r}{2a+1}\right)}{2\left(a+\frac{r}{2a+1}\right)}\right\}\left\{2a+\frac{r}{2a+1}+\frac{\frac{r}{2a+1}\left(1-\frac{r}{2a+1}\right)}{2\left(a+\frac{r}{2a+1}\right)}\right\}}{2\left\{a+\frac{r}{2a+1}+\frac{\frac{r}{2a+1}\left(1-\frac{r}{2a+1}\right)}{2\left(a+\frac{r}{2a+1}\right)}\right\}}$$

Putting $a=1$, $r=1$, we get correctly the value for $\sqrt{2}$ mentioned in Chapter II, viz.

$$\sqrt{2}=1+\frac{1}{3}+\frac{1}{3.4}-\frac{1}{3.4.34}.$$

This rule gives an approximation by defect, while the previous one gives an approximation by excess. Rodét credits this rule to the authors of the Sulvas, on the basis of geometrical devices. Bibhutibhusan Datta* is inclined to feel that the approximation was obtained by the method of continued fractions. There is no clear evidence available that the ancient Indians before Bhaskara knew anything about continued fractions, though they may have used simple examples of these. The actual proofs of the several formulæ still remain a matter for research and provide material for conjecture and speculation.

This approximation is applied in the Bakhshāli work to the calculation of error and consequent process of verification in a certain type of problem. That such a process was thought of in this early period is interesting, and the like of it is not met elsewhere. If the number of terms of an arithmetic progression whose first term is a, common difference d, whose number of terms is t, then the sum s is given in the Bakhshāli work as

$$s=\{\tfrac{1}{2}(t-1)d+a\}t. \quad \ldots \quad (1)$$

* *The Science of the Sulba.*

Conversely, given s, we have

$$t=\frac{-(2a-d)+\sqrt{(2a-d)^2+8ds}}{2d}.$$

The negative sign of the radical is not considered in the Bakhshāli work. We write this as

$$t=\frac{-p+\sqrt{Q}}{2d},$$

so that $\quad 2dt+p=\sqrt{Q}$

Also $\quad 2s=t^2d+pt.$

Often times in the examples given, $\sqrt{Q}$ does not come out exact, and a method of approximation has to be adopted. If $q_1, q_2, \dots$ are the successive approximations to the value of $\sqrt{Q}$, and $t_1, t_2, \dots,$ the corresponding values of t, these values will evidently not give s precisely, when substituted in the equation (1). The error in taking $t_1, t_2, \dots$ as the value of t is estimated in the Bakhshāli work with remarkable clarity. We have

$$2s_1=t_1^2d+pt_1$$

$$\therefore \quad 8ds_1+p^2=(2t_1d+p)^2$$

But $\quad 8ds+p^2=(2td+p)^2$

Subtracting, $\quad 8d(s_1-s)=q_1^2-Q.$

Using the approximation $\sqrt{a^2+r}=a+\frac{r}{2a}$, up to the first order, we have

$$q_1=a+\frac{r}{2a},$$

and $\quad q_1^2-Q=\left(\frac{r}{2a}\right)^2=\epsilon_1$, say.

Then ϵ_1 is the first error.

$$\therefore \quad s_1-s=\frac{\epsilon_1}{8d}.$$

Similarly, for the second approximation, the error will be

$$\epsilon_2=\left\{\frac{\left(\frac{r}{2a}\right)^2}{2\left(a+\frac{r}{2a}\right)}\right\}^2$$

and $\quad s_2-s=\frac{\epsilon_2}{8d}.$

We refer to a specific instance, in which

$$a=1,\ d=1,\ s=60.$$

The detailed workings given are

$$8ds=480,\ 2a-d=1,\ 480+1=481.$$

$$\sqrt{481}=21\frac{40}{42}=\frac{882+40}{42}=\frac{922}{42}$$

Then

$$t_1=\frac{1}{2}\left(\frac{922}{42}-1\right)=\frac{880}{84}$$

Hence

$$s_1=\frac{t_1(t_1+1)}{2}=\frac{880}{84}\times\frac{964}{168}=\frac{848,320}{14112}$$

and

$$\frac{\epsilon_1}{8d}=\frac{1}{8}\left(\frac{40}{42}\right)^2=\frac{1600}{14112}$$

$$\therefore\quad s=s_1-\frac{\epsilon_1}{8d}=\frac{848,320}{14112}-\frac{1600}{14112}=\frac{846,720}{14112}=60.$$

Again, for the second approximation

$$\sqrt{481}=21\frac{20}{21}-\frac{\left(\frac{20}{21}\right)^2}{2\left(21+\frac{20}{21}\right)}=\frac{425,042-400}{19,362}=\frac{424,642}{19,362}$$

$$\therefore\quad t_2=\frac{1}{2}\left(\frac{424,642}{19,362}-1\right)=\frac{405280}{38724}.$$

Hence

$$s_2=\frac{t_2(t_2+1)}{2}=\frac{405,280}{38724}\times\frac{444,004}{77448}=\frac{179,945,941,120}{2,999,096,352}$$

$$\therefore\quad s=s_2-\frac{\epsilon_2}{8d}=60, \text{ on simplification.}$$

The following are some interesting problems found in the Bakhshāli manuscript :

(1) Five merchants together buy a jewel. Its price is equal to half the money possessed by the first together with the moneys possessed by the others, or one-third the money possessed by the second together with the moneys of the others, or 1/4th the money possessed by the third together with the moneys of the others, or 1/5th the money possessed by the fourth together with the moneys

of the others, or 1/6th the money possessed by the fifth together with the moneys of the others. Find the price of the jewel, and the money possessed by each merchant.

If x_1, x_2, x_3, x_4, x_5 are the moneys possessed by the merchants respectively, and p the price of the jewel, we have

$$\tfrac{1}{2}x_1+x_2+x_3+x_4+x_5=x_1+\tfrac{1}{3}x_2+x_3+x_4+x_5$$
$$=x_1+x_2+\tfrac{1}{4}x_3+x_4+x_5=x_1+x_2+x_3+\tfrac{1}{5}x_4+x_5$$
$$=x_1+x_2+x_3+x_4+\tfrac{1}{6}x_5=p.$$

Hence we have

$$\tfrac{1}{2}x_1=\tfrac{2}{3}x_2=\tfrac{3}{4}x_3=\tfrac{4}{5}x_4=\tfrac{5}{6}x_5=q, \text{ say.}$$

Substituting in any of the equations above, we obtain $\frac{377}{60}q=p$. If integral solutions are required, we take $q=60m$, and then $p=377m$, where m is any integer. The answer given in the manuscript is $p=377$, x_1, x_2, x_3, x_4, $x_5=120$, 90, 80, 75, 72 respectively.

(2) Three persons possess 7 asvas, 9 hayas, and 10 camels respectively. Each gives one animal that he possesses to each of the others. They are then equally rich. Find the price of each kind of animal.

[Asva and haya both mean a horse, but the first is of superior breed].

Let x_1, x_2, x_3 be the prices of an asva, a haya and a camel respectively. Then

$$5x_1+x_2+x_3=x_1+7x_2+x_3=x_1+x_2+8x_3=M, \text{ say.}$$
$$\therefore \quad 4x_1=6x_2=7x_3=k, \text{ say.}$$

To get integral solutions, we take k as any multiple of the *L.C.M.* of 4, 6, 7. k is taken in the Bakhshāli work as $4\times6\times7=168$.

$$\therefore \quad x_1=42,\ x_2=28,\ x_3=24$$

A very similar problem occurs in Bhaskara's *Leelavati* (sl. 102), the number of persons being four, and the animals being replaced by precious stones. We have no information whether Bhaskara knew of the Bakhshāli work. It is more probable that such problems were common in the works before Bhaskara.

CHAPTER V

ARYA BHATA

30. Our knowledge of the history of Indian mathematics prior to 499 A. D. is very imperfect and spasmodic. The Jaina mathematics contained in their religious works and the Bakhshāli manuscript give us valuable information, but our knowledge even of these is very incomplete. Barring this knowledge, we are wholly in the dark as to the mathematicians and their works who lived prior to Arya Bhata. Even Arya Bhata's work known as the *Arya Bhateeya*, and written in 499 A. D. had been lost. The Indian scholar Bhau Daji got a copy of this in 1864.

A fundamental question, still not definitely answered, is about the number of mathematicians who lived by the name Arya Bhata. That there were at least two Arya Bhatas is certain : the Arya Bhata of Kusuma Pura, who wrote his work the *Arya Bhateeya* in 499 A. D., and the Arya Bhata who wrote in about 950 A. D., an astronomical treatise, the *Arya Siddhanta* or as some call it, the *Maha Arya Siddhanta*. The Muslim historian Al Biruni mentions in his history written in 1030 A. D. about two Arya Bhatas. He calls one of them as the elder Arya Bhata, and the other as the Arya Bhata of Kusuma Pura, and says that the latter has followed the doctrines of the elder. Evidently, Al Biruni was unaware of the Arya Bhata who wrote the Maha Siddhanta in about 950 A. D. One is to conclude from Al Biruni's history that there was another Arya Bhata prior to the now famous Arya Bhata. But Al Biruni's history contains so many mistakes, and inconsistencies, and his statements require careful examination. A careful examination of his references to the works of the two Arya Bhatas may lead one to come to the conclusion that all of those references have been taken from the *Arya Bhateeya* alone. But there are some facts in support of Al Biruni's statement. In the preface to his book, Arya Bhata of 950 A. D. writes

वृद्धार्य भटप्रोक्तं सिद्धांताद्यं महाकालात्
पाठैर्गतं उच्छेदं विशेषितं तन्मया स्वोक्त्या

"The Siddhanta (astronomical doctrines) given by the elder Arya Bhata is very old. On account of considerable lapse of time, the text has been subjected to many errors. I am therefore writing in my own language". But between the *Arya Siddhanta* and the astronomy contained in the *Arya Bhateeya*, there is no similarity at all, and even the fundamental doctrines differ. One has to conclude that the "elder" Arya Bhata referred to is not the Arya Bhata of Kusuma Pura, but relates to another Arya Bhata who might have lived earlier. Brahmagupta's writings may offer some corroboration to this statement. Arya Bhata of Kusuma Pura makes serious departures from some of the traditionally accepted doctrines of astronomy and propounds his own doctrines. Brahmagupta in his *Brahma Sphuta Siddhanta,* written in 628 A. D. violently criticises and falls foul of Arya Bhata for his new theories. But in his later work, the *Khandakādyaka,* Brahmagupta refers to Arya Bhata with great reverence. But the positions of the planets etc. in *Khandakādyaka* and in *Arya Bhateeya* are very different, and lead one to infer that the Arya Bhata referred to in the *Khandakādyaka* is not the Arya Bhata of Kusuma Pura. It is unfortunate that Brahmagupta has not been clear on this point. There are people who say that Brahmagupta became repentant in his old age, and has therefore given encomiums to Arya Bhata in the book written in his old age. In the presence of conflicting statements and facts, a final conclusion about the identity of the "elder Aryabhata" still awaits solution. Perhaps one day, another Arya Bhata work will come to the surface in the same way as the Bakhshāli work came to light.

31. We leave this question, and we shall now confine ourselves to the Arya Bhata of Kusuma Pura, who says that he wrote his work in his twenty-third year when the 3600th year of the Kaliyuga was running. This means that he was born in 476 A. D. and wrote his work in 499 A. D. We have said in a previous chapter that Kusuma Pura was the former name for Pātaliputra, near modern Patna. Arya Bhata belonged to the Kusuma Pura school, but it looks probable that he was a native of Kerala, in the extreme south of India. Even now the calendar based upon the system given in the *Arya Bhateeya* is followed to some extent in Kerala.

The *Arya Bhateeya* is a very small work. Brahmagupta in his *Brahma Sphuta Siddhanta* divides the work into two parts. Barring the three *slokas* (verses) forming the introduction and the conclusion, there are ten *slokas* written in the *Geetika* metre, followed by 108 *slokas* in the *Aryavritta* metre, Brahmagupta calls these two parts as the *Dasageetika* (दशगीतिका) and the *Aryāshta shatam* (आर्याष्टशतम्) respectively. Arya Bhata himself gives only the name *Arya Bhateeya* to his whole work. The *Aryashta Shatam* comprises three divisions: *Ganita* or mathematics, *Kāla Kriya* (calculation of time) and the *Gōla* (sphere). The last division really refers to the celestial globe, and astronomical terms and calculations.

The book is very concise, and there is considerable difficulty in understanding the meaning in several places. There might have been a *Karana Grantha* or commentary, providing detailed explanations. Or, it may be that Arya Bhata just wrote a small work adding his own discoveries or theories to the then existing knowledge. The subdivision by name *Ganita* deals with pure mathematics, and in all contains 33 slōkas only. The subjects dealt with are the methods for determining square and cube roots, some geometrical problems, the progression, problems involving quadratic equations, and lastly indeterminate equations of the first degree. The method of solving these equations has been called ***Kuttaka*** **by later mathematicians.** After the Jaina works and the **Bakhshāli work,—all** prior to Arya Bhata—came to light, **the portions of Arya Bhata's** work pertaining to geometry, the progressions **and some other topics** have lost their importance. It is now **clear that Arya Bhata has** simply reproduced known knowledge about these topics. **The main** topics in pure mathematics which stand to the credit of Arya Bhata today are his value of π and his table of sines, besides his method of solving indeterminate equations of the first degree. Before we deal with these topics, we may say a few words about a system of enumeration which may be called *word numerals* and alphabet numerals, of which the latter system was introduced for the first time by Arya Bhata.

32. The method of denoting certain numbers by means of certain words is called the system of word numerals, and is a system prevalent even today in the poetry of Indian languages. Thus sun

means twelve, eyes = 2, agni = 3, vedas = 4, bāna = 5, samudra (sea) = 7, kari (elephant) = 8, rudra = 11, rāja (king) = 16. The numbers relating to these words are based upon Indian mythology. The system was probably first introduced by the famous astronomer Varāha-mihira (505 A. D.) and is entirely absent in the *Arya Bhateeya*.

Alphabet numerals is a different and more comprehensive system, which was in vogue in some countries before the decimal system of numeration was introduced. Thus in ancient Greece, there was the system $A=1$, $B=2$, $\Gamma=3$, $\Delta=4$, etc. But the origin of the alphabet numeral system in India was not on account of the absence of a satisfactory system of numeration, but because it was helpful in poetry. It will be noted that practically all the ancient writers on mathematics have given their rules or theorems in the form of poetry. Thus to denote the number 1234565432l, Mahāvira calls it as ēkadishadānthāni kramēna heenāni (ऽकादिषदांतानि क्रमेण हीनानि) i.e. "going from 1 to 6 and steadily coming down." Arya Bhata in his *Dasa Geethika* introduces a scheme which is refreshingly original though complicated. There are 25 vargeeya (classified) consonants and 9 avargeeya (unclassified) consonants in the Devanāgari or Sanskrit script, besides 10 vowels. The 25 vargeeya consonants are given the numbers 1 to 25, the 9 avargeeya consonants *ya* (य), *ra* (र), *la* (ल), *va* (व), *sha* (श), *Ṡha* (ष), *sa* (स), *ha* (ह), *ḷa* (ळ) are made to denote 30, 40, 50 etc. The vowels indicate powers of ten in order. The following examples from the *Arya Bhateeya* will illustrate how any number can be expressed in terms of the letters of the alphabet : *khya* (ख्य) = ख + य = 2 + 30 = 32, *khyu* = ख + य + उ = 32,000, *ghr* (घृ) $= 4 \times 10^6$, *khyūghr* (ख्यूघृ) = 4,320,000, *cha ya gi yi ngu shu chchlr* (च य गि यि ङु शु छ्लृ) $= 6 + 30 + 300 + 30 \times 100 + 5 \times 10{,}000 + 70 \times 10{,}000 + (7+50) \times 1{,}000{,}000 = 57{,}753{,}336$.

The Hindu method of expressing everything (even mathematics) in poetic form must be noted in order to appreciate this exceedingly queer, if original, method of numeration. It is not on account of want of knowledge of the decimal system, as is argued by unfriendly critics, that this system was introduced. On the other hand, the method affords a proof that the decimal system was well in vogue. This is of course amply established by the methods for square and

cube roots of large numbers that Arya Bhata expounds in his work, —methods which are practically identical with our present methods and which are impossible without the decimal system.

33. It is now certain that Arya Bhata's method for extracting the square root of a number was not his own, since we find in the Jaina mathematics the extraction of the square roots of very large numbers. The actual working rule is not explained in Jaina mathematics, and Arya Bhata gives this in clear form, which is precisely the present form.

As regards cube roots, we have so far no evidence of the method having been known earlier. Arya Bhata's rule is explained below, as applied to a concrete example.* The signs o and — denote what Arya Bhata calls as *ghana* (घन) and *aghana* (अघन) places — cube and non-cube places.

	$\overset{\circ}{1}\ \bar{8}\ \bar{6}\ \overset{\circ}{0}\ \bar{8}\ \bar{6}\ \overset{\circ}{7}$ (root = 1
cube of root	1
Three times square of root = 3)	08(2 = quotient or next digit of root
i.e. 3×1^2	6
square of quotient multiplied by three times the pūrva (previous)	26
i.e. $2^2 \times \ \times 1$	12
	140
cube of quotient	8
Three times square of root 432)	1328(3 = quotient or next digit of root
$= 3 \times 12^2$	1296
sq. of quotient multiplied by three times the pūrva	326
i.e. $3^2 \times 3 \times 12$	324
	27
cube of quotient	27
	0

cube root is 123.

* The example is taken from W. E. Clark : *Arya Bhhteeya* (Chicago University Press), 1930.

34. Problems on arithmetic progressions dealt with in the *Arya Bhateeya* need not be discussed here, since these figure in the Jaina and Bakhshāli mathematics. But the following identities occur for the first time, in Arya Bhata's work :

$$1^2+2^2+\ldots\ldots+n^2=\tfrac{1}{6}n(n+1)(2n+1)$$

$$1^3+2^3+\ldots\ldots+n^3=(1+2+\ldots\ldots+n)^2$$

$$=\tfrac{1}{4}n^2(n+1)^2$$

$$1+(1+2)+(1+2+3)+(1+2+3+4)+\ldots\ldots \text{ to } n \text{ terms}$$

$$=\tfrac{1}{6}n(n+1)(n+2), \quad \text{or} \quad \tfrac{1}{6}\{(n+1)^3-(n+1)\}.$$

The results in Arya Bhata's work relating to circles including the formula for the area of a circle are evidently taken from the earlier Jaina works. Arya Bhata wrongly gives the volume of a sphere of radius r as $\pi r^2\sqrt{\pi r^2}$, and the volume of a triangular pyramid as area of the triangle $\times$ height $\times \frac{1}{2}$.

35. The familiar approximation $\pi=3.1416$ was first given by Arya Bhata in the form that the circumference of a circle whose diameter is 20,000 is 62, 832. The text is as follows :

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथासहस्राणाम्

अयुत द्वय विष्कंभस्यासन्नो वृत्त परिणाह:

The word आसन्न = approximate should be noted. We shall comment in a later chapter (Chap. XII, §108) as to how Arya Bhata not only took the above value for π as approximate, but also he probably had an inkling into the fact that π is irrational.

This value of π as also his table of sines discussed below must have been obtained geometrically by divisions of a quadrant of a circle. The value $\pi=3.1416$ has been used (necessarily in the form of a fraction, since decimals were not known in India) by later writers like Varāha Mihira, Lalla and Bhaskara II. Bhaskara II writes it as $\frac{3927}{1250}$. There is no doubt that all these later writers have taken it from Arya Bhata, though Arya Bhata himself might have taken it from the earlier *Surya Siddhantas* which have now been lost. The value $\pi=\sqrt{10}$ was also in general use for a long time, just in the same way as we use $\frac{22}{7}$ even now in preference to 3.1416 for rough calculations. The use of the value $\sqrt{10}$ will not afford a

"proof" for the concocted theory set forth by unfriendly critics that the value 3.1416 for π is of Greek origin. Archimedes gave the value of π as lying between $3\frac{1}{7}$ and $3\frac{10}{71}$. The value $\frac{22}{7}$ is the Archimedean value, and no other Greek has given any different value.

36. We now come to the table of sines whose construction presents a problem which remains rather unsolved. Whether Arya Bhata himself constructed this table, or whether he took it from a *Surya Siddhanta* which was extant in his time, will remain a matter of speculation. We first explain the Hindu meaning of sine.

In the figure, PAP' is an arc of a circle, and A its middle point. O is the centre of the circle. On account of the shape, PAP' is called the bow (*chāpa* or *dhanus*), and the chord PNP' is the bow-string or rope, called *jya* (ज्या) in Sanskrit. In the course of time, NP which is half this *jya* itself came to be called as *jya*. This *jya* itself is the Indian sine, so that if $A\hat{O}P=\theta$, *jya* $\theta=NP=r\sin\theta$ where r is the radius. In other words, the Indian sine is the modern sine multiplied by r. It was the practice to use a specific value for r. It will be interesting to observe how the word *jya* has degenerated into the word 'sine'. An alternative word for *jya* viz. *jīva* became *jība* at the hands of the Arabs, and later it became *jaib*. There exists an Arabic word with similar pronunciation which means heart. Later at the hands of the Romans, these two words were interchanged by mistake, and so the *jya* became *sinus* (=bosom, heart). Hence came the word sine, providing an extreme example of a mathematical term which is completely bereft of its etymological meaning.

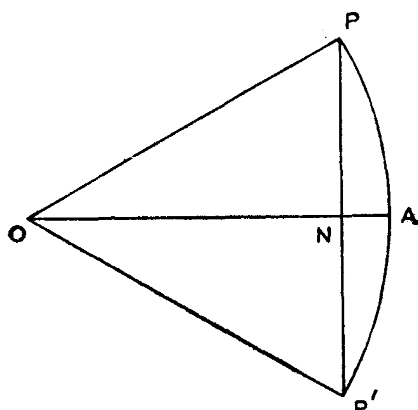


FIG. 7.

To choose r, the circumference of a circle was divided into 360×60 or 21600 equal parts. If the length of arc of each one of these parts is to be unit length, the radius should be $21600/2\pi$. Taking Arya Bhata's value for π, this gives $r=3437.7$ or 3438* to the nearest integer. Arya Bhata's table of sines is constructed, taking $r=3438$. The table gives only the sines of multiples of $3\frac{3}{4}°$. The sines of other angles between 0° and 90° were obtained by interpolation, by applying the method of proportion to the differences.

Since 3°45′ or 225′ is a small angle, they took the arc as equal to the chord for this small angle, and took *jya* 3°45′=225. The following table, gives the *jya* numbers of multiples of 3°45′, according to the *Surya Siddhanta*.† The *Arya Bhateeya* which is brief gives only the differences in the tenth slōka of the first part (*Dasageetika*). The method of constructing these differences is enunciated in the *Ganita Pada*, sl. 12, without adequate explanation. From these differences, one can of course construct the actual sines, taking the first sine as 225.

The values of sines or *jya* and *utkrama jya* (उत्क्रम ज्या) or $r-\cos\theta$ as given in the *Surya Siddhanta* are given in the table. The values of $r-\cos\theta$ are obtained by subtracting from 3438 the sines taken from the bottom, in the reverse order. This means that the formula $\cos\theta=\sin(90°-\theta)$ has been recognised and used. For the sake of comparison, $3438\sin\theta$ is given in the third column, as calculated from the modern table of sines. The comparison between the second and third columns is very close. Nowhere is the difference between the modern sine and $\frac{1}{3438}$ of the Hindu sine more than 1 in 3438 i.e. more than 0·0003. The comparison is much closer in several places.

The last column gives the differences between consecutive sines given in the second column. As stated before, these are the "sines" given by Arya Bhata in the tenth *slōka* of the *Dasa Geethika*. By "sines", we have to understand the differences between the sines.

* To get 3438 exactly, we must have $\pi=600/191$, which is the value of π given by Arya Bhata the second (third ?) about 950 A. D.

† *Surya Siddhanta* (English Translation by Burgess), Calcutta University Press, Ch. II, stanzas 15-27, pp. 58-59.

Angle θ	*jya* or Hindu sine	$(\sin\theta) \times 3438$	Hindu versine $r - jya\,(90° - \theta)$	Hindu sine differences
3°45′	225	224·85	7	224
7°20′	449	448·95	29	222
11°15′	671	670·72	66	219
15°0	890	889·82	117	215
18°45′	1105	1105·01	182	210
22°30′	1315	1315·05	261	205
26°15′	1520	1520·58	354	199
30°0′	1719	1719·00	460	191
33°45′	1910	1910·05	579	183
37°30′	2093	2092·09	710	174
41°15′	2267	2266·08	853	164
45°0	2431	2431·01	1007	154
48°45′	2585	2584·08	1171	143
52°30′	2728	2727·55	1345	131
56°15′	2859	2858·55	1528	119
60°0′	2978	2977·04	1719	106
63°45′	3084	3083·45	1918	93
67°30′	3177	3176·06	2123	79
71°15′	3256	3255·75	2333	65
75°0′	3321	3320·85	2548	51
78°45′	3372	3371·95	2767	37
82°30′	3409	3408·75	2989	22
86°15′	3431	3430·85	3213	7
90°0′	3438	3438	3438	

37. The fundamental question is as to how these differences or "sines" were obtained. The following rule is found in the *Surya Siddhanta* (ch. 2, st. 15—16).

राशि लिप्ताष्टमो भागः प्रथमं ज्याऽर्धमुच्यते
तत्तद्विभक्तलब्धोन मिश्रितं तद्द्वितीयकं
आद्ये नैवं क्रमात् पिंडान् भक्त्वा लब्धोन संयुताः
संज्ञकाः स्युश्चतुर्विंशा ज्यार्धपिंडाः क्रमादमी

Rasi means $30°$. Its eighth part $=3\frac{3}{4}°=225'$. This is the first *jya*, i.e. jya $225'=225$. This is the meaning of the first line. Divide this by itself, subtract the quotient from the same (225), and add the result to it. We thus get the second *jya*, i.e. *jya* $7°30'=225+(225-\frac{225}{225})=449$, In this way, divide the successive *jyas* by 225, substract from 225 the sum of the quotients, and add the remainder to the last *jya*. We thus obtain the next *jya*. Put in terms of formulæ, let $s_1, s_2, \ldots, s_{24}$ be the *jya* numbers and $q_1, q_2, q_3, \ldots,$ quotients when these are divided by 225. Then we have

$$\begin{aligned} s_1 &= 225 \\ s_2 &= s_1+225-q_1 \\ s_3 &= s_2+225-(q_1+q_2) \\ \ldots & \quad \ldots \quad \ldots \quad \ldots \\ s_r &= s_{r-1}+225-(q_1+q_2+\cdots+q_{r-1}), \qquad \text{(A)} \\ & \qquad r=1, 2, \ldots, 24. \end{aligned}$$

The rule in the *Arya Bhateeya* is as follows :

प्रथमाच्चापज्यार्धाद्यैरूनं खंडितं द्वितीयार्धं
तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि

(Ganita Pāda, sl. 12)

The style of the language is complicated and condensed, and the meaning is not clear. S. N. Naraharayya* explains that the meaning of this stanza is essentially the same as the stanza from *Surya Siddhanta*. To get this interpretation, he interprets प्रथम (Prathama) to mean शेष (remainder) and द्वितीयार्धं=द्वितीयज्यार्धं. He

* *Jour. Indian Math. Soc.* 15 (1924—25), second part, Notes and Questions, p. 113.

translates the *sloka* thus : "The first sine being divided by itself and the quotient obtained, the same (i.e the first sine) together with the same minus the quotient is the second sine ; and the other sines are obtained by successively subtracting the sum of all the quotients from the first sine and adding the results successively to the last of the already obtained sines". Symbolically,

$$s_r = s_{r-1} + s_1 - \sum_{i=1}^{r-1} q_i.$$

A. A. Krishnaswami Ayyangar* interprets the *sloka* in a different way.

"Each of the sine-differences is less than the first sine by (two quantities) the difference between the first sine and the preceding sine-difference as well as by the quotient obtained by dividing (the sum of) all the preceding sine-differences by the first sine".

Let Δ_1 = Arya Bhata's sine of $3\frac{3}{4}°$ = the first sine, and $\Delta_2, \Delta_3, \Delta_4, \ldots$ the successive sine-differences. Then the meaning of the rule is equivalent to the formula

$$\Delta_{r+1} = \Delta_1 - (\Delta_1 - \Delta_r) - \frac{\Delta_1 + \Delta_2 + \ldots + \Delta_r}{\Delta_1}$$

i.e. $$\Delta_r - \Delta_{r+1} = \left(\sum_1^r \Delta_r\right) \Big/ \Delta_1,$$

which can also be written

$$\sin(r+1)\alpha - \sin r\alpha = \sin r\alpha - \sin(r-1)\alpha - \frac{\sin r\alpha}{225}.$$

Arya Bhata in his *Dasa Geetika* calls the sine differences themselves as sines. Krishnaswami Ayyangar might have got the above interpretation from this fact, with a further interpretation that *dvitiya* द्वितीय means "previous" instead of the usual meaning "second".

Since both interpretations lead to the same formula marked (A) above, we need not discuss which is the true translation. It is fairly certain that these rules were formulated as a consequence of geometrical calculations taking a quadrant of a circle, and angles of multiples of $3\frac{3}{4}°$. We give below the calculations as expounded by the two writers quoted above, in order.

* *Juor. Indian Math. Soc.*, 15 (1924—25), First part, 121—6.

(1) Divide the arc XY of a quadrant of a circle of radius 3438 into 24 equal parts at the points P, Q, R etc. Draw PR_1, QR_2, RR_3, ... perpendiculars to XY. These are in order the first, second, third etc. *jyas*. Draw the tangents at P, Q, R, ... to meet OX, OP, OQ, ... at X', P', Q', ...

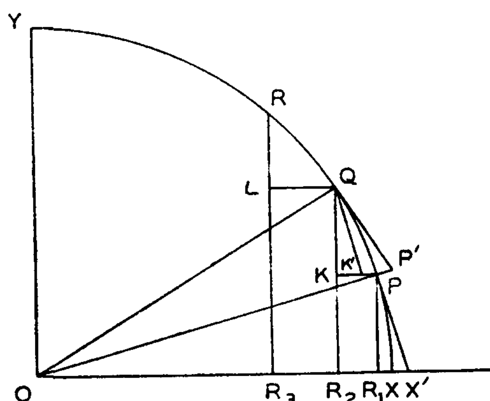


FIG. 8.

Draw PK, QL, ... perpendiculars to QR_2, RR_3, ... Then QK, RL, ... are the successive differences of the sines. Draw also QK', RL', SM', ... (not shown in the figure) parallel to PX', QP', RQ', ...

Then, $R_1X = OX - OR_1 = OX - \sqrt{OP^2 - PR_1^2}$

But $PR_1 = PX$ approximately $= 225$

$\therefore \quad R_1X = 3438 - \sqrt{3438^2 - 225^2} = 7$, to the nearest integer.

X is approximately the middle point of R_1X'.

$\therefore \quad R_1X' = 14$.

$\triangle QKK'$ may be considered as the displaced position of $\triangle PR_1X'$.

$\therefore \quad KK' = R_1X' = 14$, and $K'P = 7$ (being approximately taken as the displaced position of XX'). Hence $KP = 21$.

$\therefore \quad QK = \sqrt{PQ^2 - KP^2} = \sqrt{225^2 - 21^2} = 224$.

In a similar manner, $LQ = 5 \times 7 = 35$ and $RL = \sqrt{225^2 - 35^2} = 222$.

These lead to the formulation of the differences Δ_1, Δ_2, ...

$$\Delta_1 = 224 = 225 - 1 = s_1 - q_1$$

$$\Delta_2 = 222 = 225 - (1+2) = s_1 - (q_1 + q_2), \text{ and so on.}$$

Let α denote the angle $3\frac{3}{4}°$. On the quadrantal arc AB mark P, Q, R corresponding to the angles $AQP=(r-1)\alpha$, $AOQ=r\alpha$, $AOR=(r+1)\alpha$. Let PR cut OQ at S, and P', Q', R', S' be the projections of P, Q, R, S on OB.

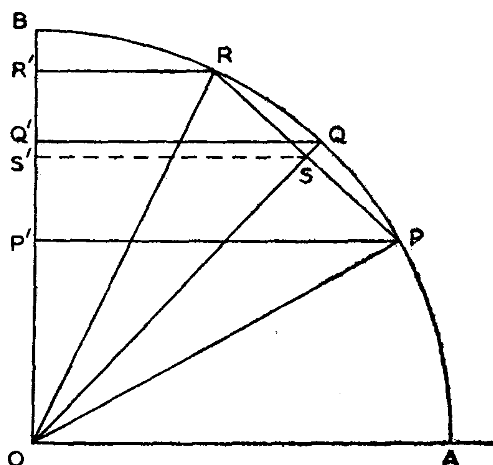


FIG. 9.

Then, $P'S'=S'R'$, since $PS=SR$.

$$\sum_1^r \Delta_r = OQ',\ \Delta_{r+1}=Q'R',\ \Delta_r=P'Q'.$$

$$\therefore\quad \Delta_r-\Delta_{r+1}=2S'Q'=2\frac{SQ}{OQ}.OQ',\ \text{since } \frac{S'Q'}{SQ}=\frac{OQ'}{OQ}=2\frac{SQ}{OQ}.\sum_1^r \Delta_r.$$

If the tangent at R meets OQ in Q', $2SQ=SQ'$, approximately. From the right-angled $\Delta ORQ'$,

$$Q'S.Q'O=Q'R^2$$

$$\therefore\quad \frac{2SQ}{OQ}=\frac{QR^2}{OQ^2}, \text{ approximately}$$

$$=\left(\tfrac{225}{3438}\right)^2=\tfrac{1}{225}$$

$$\therefore\quad \Delta_r-\Delta_{r+1}=\sum_1^r \Delta_r/225.$$

The proof in (2) is more convincing than that in (1). All that we can safely say is that Arya Bhata must have based his rule or some such geometrical calculations, and the assumption of a number of approximations.

38. The values of the *jyas* calculated on the basis of the above formula do not quite agree with those given in the table. To remove the deviations, certain subsidiary rules have been formulated.

(i) In calculating the q's, the usual convention to drop the fractional part if it is less than $\frac{1}{2}$ and to take the next integer if it is more than $\frac{1}{2}$ is to be adopted in general, subject to some exceptions. By rigidly following this convention, certain excesses might continually accumulate or diminish, thus interfering with the accuracy of the results. Hence a special convention is introduced for the calculation of some of the q's.

(ii) एकविंशाच्च विंशाच्च षष्ठास्तं च दशादपि

सप्तमात् द्वादशात्सप्तदशानर्धोत्तरं मतम्

(Brahmasphutā siddhanta, cited by Ranganatha)

i.e. the fractional part though greater than half should be dropped, and only the integral part retained in the case of q_6, q_7, q_{12}, q_{15}, q_{17}, q_{20} and q_{21}.

(iii) Further, diminish $\sum q_r$ by

(a) 1, 2, 3, 4, 5 respectively, before subtracting from s_1, when $r=7, 8, 9, 10, 11$

(b) by 7, 9, 11 respectively when $r=12, 13, 14$

(c) by 14, 17, 20 respectively when $r=15, 16, 17$

(d) by 25, 30 respectively when $r=18, 19$

(e) by 36, 43, 50, 58 respectively when $r=20, 21, 22, 23$.

It is difficult to understand the basis of these subsidiary rules, and it leaves us in some doubt as to the actual method or details that Arya Bhata (or the authors of the *Surya Siddhanta*) have followed in the construction of the table of sines.

While further investigation may be necessary to throw more light on the above, we need not take seriously the biassed view that the Hindu sine-table owes its origin to the Greek table given by Ptolemy. The two tables differ in some places, and the Greek table cannot explain the rule formulated above and the subsidiary corrections to it.

39. Perhaps the most important contribution of Arya Bhata in the field of pure mathematics is his solution of the indeterminate equation $ax-by=c$. We shall deal with this at length in Chapter IX,

discussing also the improvements and modifications suggested by later writers.

Arya Bhata is the first known Indian astronomer, since little or nothing is known about the authors of the *Surya Siddhanta* and other astronomical works prior to Arya Bhata. Arya Bhata was the first astronomer who mentioned that the diurnal motion of the heavens is due to the rotation of the earth about an axis. This doctrine was not generally accepted by Indian astronomers, and Arya Bhata came in for severe criticism at the hands of Brahmagupta.

40. We include in this chapter a brief mention of Varāha Mihira, who was mainly an astronomer and astrologer. Varāha Mihira (the second scientist to bear this name, nothing, however, being known about the first), son of Āditya Dāsa of Kapiththaka wrote in 505 A. D. his great work *Pancha Siddhantika,* which expounds the astronomical doctrines contained in the five treatises after the names of Paulisa, Romaka, Vāsishta, Soura and Brahma. Varāha Mihira says that of all these, the Soura system is the only correct one. Varāha Mihira's work was discovered and resuscitated through the efforts of Pandit Sudhākar Dvivedi and Dr. Thibaut. Varāha Mihira also wrote his famous astrological treatises *Brihajjātaka* and *Brihat Samhita*. These also include the astronomical topics relating to the determination of time and of planetary position, and of eclipses.

Varāha Mihira's work reveals that the correction known as the precession of equinoxes was known in his time, and a fairly accurate value was available. He prepared a corrected version of the Indian calendar, after taking into account the amount of precession that had accumulated since the preparation of the *Surya Siddhanta*. He also left a warning for the future calendar-makers that the calendars have to be periodically corrected by taking into account the accumulated precession—a warning which has not been heeded to properly, till it was taken up recently by the Government of India which took steps to form a National calendar.

REMARKS ON THE HINDU SINE-TABLE

In his thesis for the Doctorate Degree of the Karnatak University, my student Dr. D. A. Somayaji draws attention to Bhaskara's commentary under verses 1—25 of *Jyotpathi vasana* in his *Goladhyaya*, according to which the table of 24 sines was constructed by the help of the two formulæ

$$H\sin^2\theta + H\cos^2\theta = R^2 \qquad \text{(a)}$$

$$H\sin\frac{\theta}{2} = \frac{1}{2}(H\sin^2\theta + H\,\text{vers}^2\,\theta)^{\frac{1}{2}}. \qquad \text{(b)}$$

Here $H\sin\theta$ denotes the Hindu sine of θ, which is equal to the modern sine multiplied by R, the radius of the circle used to define the Hindu sine. Similarly for $H\cos\theta$, etc. Bhaskara has outlined the method of proof of (b).

Calling the 24 H sines as H_r, $r=1, \ldots, 24$, and taking $R=3438$, we know

$$H_8 = H\sin 30° = 1719 \qquad H_{12} = H\sin 45° = \frac{1}{\sqrt{2}}R = 2431$$

$$H_{16} = H\sin 60° = \frac{\sqrt{3}}{2}R = 2977 \qquad H_{21} = H\sin 90° = 3438.$$

Now, by the help of (b), H_4 is obtained from H_8, and H_2 is obtained from H_4. Similarly H_{12} leads to the values of H_6 and H_3. Then, by the help of (a), we obtain H_{20}, H_{22}, H_{23}, H_{18} and H_{21}. Again, from H_{20}, H_{22} and H_{18}, we obtain H_{10}, H_{11} and H_9 by the help of (b), and H_5 from H_{10}. Then (a) leads to the values of H_{14}, H_{13}, H_{15} and H_{19}. Then H_{14} leads to H_7 and H_7 leads to H_{17}. Thus the table of 24 sines is completed.

Bhaskara gives more improved methods to construct the sine table when the quadrant is divided into 30 equal parts, as also a table of 90 sines. It is in this context that he was led to formulate the first example of a differential coefficient, viz.

$$\delta \sin x = \cos x\, \delta x.$$

Dr. Somayaji is of the opinion that it is by this method that Arya Bhata and his predecessors constructed the sine table. His statement that this method of Bhaskara was known to Arya Bhata and his predecessors requires to be substantiated. The method that we have detailed in Chapter V is based on finite differences, and is therefore altogether different. The method has been substantiated on the strength of a rule found in the *Surya Siddhanta*. *The use of finite differences at so early a period is a very important observation that demands attention.* If Dr. Somayaji's statement is correct, this may imply that the H sines were calculated by both the methods, and then while trying to explain the small discrepancies in some cases, they introduced the subsidiary corrections referred to (§38). This may give an easy and plausible explanation for these subsidiary corrections, for which no satisfactory explanation has so far been found.

Chapter VI

BRAHMAGUPTA

41. After Varaha Mihira, the most celebrated mathematician belonging to the Ujjain school is Brahmagupta. He was born in 598 A. D. and was probably a native of Sind, now part of W. Pakistan. According to his own statement, his father was Jishnu, and he wrote his *Brahma Sphuta Siddhanta* in his thirtieth year in the reign of Shri Vyaghra Mukh of the Saka dynasty. He has given that name to his work, because he corrected and brought up-to-date the old astronomical work, *Brahma Siddhanta*. The book is a compendious volume on astronomy of which $4\frac{1}{2}$ chapters are devoted to pure mathematics. Brahmagupta has called the twelfth chapter as *Ganita* (arithmetric) and the eighteenth chapter as *Kuttaka*. According to the tradition of those times, *Ganita* includes matter pertaining to arithmetic as also the progressions and a few geometrical topics. The word *Kuttaka*, which literally means pulverizer, may be compared with the word *analysis*, and denotes algebra. It is not definite when the well-known word *Bīja Ganita* became to be used to connote algebra. It occurs for the first time in the writings of Prithūdaka Swami (*c.* 860 A. D.), the celebrated commentator of Brahmagupta. From the time of Bhāskara, the word *Kuttaka* has been used to denote the solution of the indeterminate equation $ax-by=c$ only.

It was through the *Brahma Sphuta Siddhanta* of Brahmagupta that the Arabs became conversant of Indian astronomy. A famous king by name Khalif Abbasid Al Mansoor (712-775) founded on the banks of the Tigris the city of Baghdad, and made it a centre of learning. At his invitation, a scholar of Ujjain, by name Kanka came to Baghdad by about 770 A. D. and explained to the Arabs the Hindu system of arithmetic and astronomy. By the Khalif's orders, the *Brahma Sphuta Siddhanta** that Kanka was using was

* Hereafter, we shall abbreviate this by writing B. S. S.

translated by Al Fazãri into Arabic, and was named *Sind Hind* or *Hind Sind.* Al Fazãri's work was in general use among the Arabs for a long time. An abridged edition was published by Muhammed ben Musa. The latter also worked out another compilation of tables based on the Indian and Persian tables as also on Ptolemy's astronomical work. In his translation of Al Biruni's *India*, Dr. Sachau remarks* that the Arabs learnt their astronomy from Brahmagupta's work, before they came to know about Ptolemy. Besides the *Sind Hind*, there was one other translation of Indian astronomy, by name *Alarkand* which was known to the Arabs. This may have been a translation of *Surya Siddhanta* ; *Arka* (अर्क) in Sanskrit means Sun or *Surya* (सूर्य). The *Arya bahar* (*Arya Bhateeya*) does not seem to have reached the Arabs.

42. As was usual with the ancient Indians, all matter is contained in poems. In about 860 A. D., Prithūdaka Swami wrote an important commentary on the B. S. S., which contains many examples to illustrate the mathematical statements and results contained in the B. S. S. There is considerable difference of opinion as to whether these examples are those of Brahmagupta or of Prithūdaka Swamy. The examples are not found in any text of the B. S. S., but this alone is insufficient to assert that the examples are those of the commentator. Writing materials were scarce in those days, and students would learn at the feet of their masters, and pass that knowledge in turn to their students. In this way, some or all the examples might have come down to Prithūdaka Swamy. A mathematician of Brahmagupta's calibre would not just write down a set of mathematical rules, without illustrating them by suitable examples.

43. The most outstanding contribution of Brahmagupta to mathematics is his solution of the indeterminate equation $Nx^2+1=y^2$. We shall disscuss this fully in a later chapter, explaining Bhaskara's contribution also to that subject. We shall confine ourselves in this chapter to his smaller contributions to algebra and geometry. The chapters pertaining to arithmetic contain nothing new, since all the

* Vide Coletrooke's *Introduction* to his "Algebra with Arithmetic and Mensuraion from the Sanscrit of Brahmagupta and Bhascara" (1817).

topics have been covered by previous writers. We meet in *Arya Bhateeya* itself problems on simple interest which involve quadratic equations. We have the following problem in Brahmagupta : The interest accruing on 500 for 4 months is lent out at the same rate for 10 months, and the amount is 78. What is the rate of interest ?

If x is the interest on 500 for 4 months, this gives the equation $\frac{x^2}{200}+x=78$ $\therefore$ $x=60$, or the rate of interest is 3% per month.

44. *Algebra.* (1) To evaluate $\frac{a}{b}$, the following rule is given (xii, sl. 58), and it will be useful to remember this rule :

$$\frac{a}{b}=\frac{a}{b+h}+\left(\frac{a}{b+h}\right)\frac{h}{b}.$$

Thus, to evaluate $\frac{1920}{93}$, take $h=3$.

$$\therefore \quad \frac{1920}{93}=\frac{1920}{96}+\frac{1920}{96}.\frac{3}{93}=20\frac{60}{93}.$$

To evaluate $\frac{9999}{97}$, take $h=2$

$$\therefore \quad \frac{9999}{97}=101+\frac{202}{97}.$$

Applying the rule again, $\frac{9999}{97}=101+2+\frac{8}{97}=103\frac{8}{97}$.

(2) We have

$$x^2=(x-y)(x+y)+y^2. \quad \text{[xii, sl. 63]}$$

Thus, $77^2=74.80+9=5929$.

This result was known to the Greeks, and is known as the formula of Nichomachus. Its geometrical aspect and proof are evident.

(3) The sum of a geometric progression. The formula

$$a+ar+ar^2+ar^3+\ldots \text{ to } n \text{ terms}=\frac{a(r^n-1)}{r-1}$$

was not in vogue with the Indians. Instead, they were arriving at the value of r^n by a process akin to the rule of practice in arithmetic. The following rule is given by Prithūdakaswami, and it may have been his own or of his master Brahmagupta.

If n is even, write $\frac{n}{2}$ in one place, and write s (to mean square) by its side ; if n is odd, write $n-1$ and m (to mean multiply) by its side. Repeat the same process for $\frac{n}{2}$ or $n-1$ as the case may be, according

as this number is even or odd. Continue this process, till the end (i.e. till the number 1 is obtained). Now, commencing from the bottom, proceed upwards multiplying by r at places marked m, and squaring the previous expression at places marked s. If N is the number finally obtained, then the sum of the series is $\frac{a(N-1)}{r-1}$.

One or two illustrations will explain the process. It will be seen that in all cases, $N=r^n$.

Ex. 1.	$n=31$		*Ex. 2.*	$n=24$	
$31-1$	m	r^{31}	$\frac{24}{2}$	s	r^{24}
$\frac{30}{2}$	s	r^{30}	$\frac{12}{2}$	s	r^{12}
$15-1$	m	r^{15}	$\frac{6}{2}$	s	r^{6}
$\frac{14}{2}$	s	r^{14}	$3-1$	m	r^{3}
$7-1$	m	r^{7}	$\frac{2}{2}$	s	r^{2}
$\frac{6}{2}$	s	$(r^3)^2$	1		r
$3-1$	m	$r^2\times r$		$N=r^{24}$	
$\frac{2}{2}$	s	r^2			
1		r			
	$N=r^{31}$.				

45. *Geometry.* A triangle whose sides a, b, c are connected by the relation $a^2+b^2=c^2$ is right-angled. The triangle with sides 3, 4, 5 has come down from most ancient times. We have referred to other such examples in the chapter on Sulva Sutras (§11). Brahmagupta interested himself in the determination of rational right-angled triangles i.e. right-angled triangles whose sides are rational numbers, and similarly of rational cyclic quadrilaterals. For the former, Brahmagupta gives the general solution without proof, viz.

$$a=2mn, \qquad b=m^2-n^2, \qquad c=m^2+n^2$$

where m and n are any unequal rational numbers. (xii, sl. 33).

A number of modifications of this problem present themselves, as follows :

(a) Given a side a or b, to construct a right-angled triangle with rational sides.

Brahmagupta's solution (sl. 35) is a, $\frac{1}{2}\left(\frac{a^2}{m}-m\right)$, $\frac{1}{2}\left(\frac{a^2}{m}+m\right)$, where m is any rational number. Bhaskara and Mahavira give the solution

$a, \frac{2na}{n^2-1}, \left(\frac{n^2+1}{n^2-1}\right)a$, while Karavindaswami gives

$$a, \left(\frac{n^2+2n}{2n+2}\right)a, \left(\frac{n^2+2n+2}{2n+2}\right)a.$$

These solutions are simple transformations of Brahmagupta's result. Putting $m=\left(\frac{n-1}{n+1}\right)a$ in Brahmagupta's result, we get the Mahavira-Bhaskara result, and putting $m=\frac{a}{n+1}$, we get Karavindaswami's result.

(b) Given the hypotenuse c, to construct a rational right-angled triangle.

We may mention this here, although Brahmagupta does not deal with this problem. Mahavira gives the solution $c, \frac{2mnc}{m^2+n^2}, \left(\frac{m^2-n^2}{m^2+n^2}\right)c$, while Bhaskara's solution is $c, \frac{2px}{p^2+1}, \left(\frac{p^2-1}{p^2+1}\right)c$. This readily follows from Mahavira's solution by putting $\frac{m}{n}=p$.

Both the problems (a) and (b) are readily adapted* from the general solutions $a=2mn$, $b=m^2-n^2$, $c=m^2+n^2$. Dividing throughout by $\frac{a}{2mn}$, and then putting $\frac{an}{m}=p$, we obtain the solution in (a). Divining by m^2+n^2, we get the solution in (b). Dickson† is evidently unaware of the works of Mahavira and Brahmagupta, when he ascribes the results in (a) and (b) to Leonardo Fibonaççi (1202), and Vieta (1580).

46. The following problem will be a corollary to the above.

To construct a triangle in which two sides and the altitude through their point of intersection are rational numbers.

This is readily done by constructing two right-angled triangles having any given rational number x as a side, and placing the triangles side by side in opposite senses, as in Fig. 10. Then the two triangles combined give the required triangle.

* G. S. S. of Mahavira, edited by M. Rangacarya (1912), vii, 112½, 222½.

† Dickson. *History of Theory of Numbers*, Vol. II, pp. 192—3.

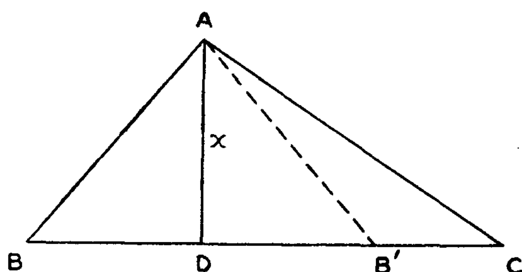


FIG. 10.

By §45(a), $AB=\frac{1}{2}\left(\frac{x^2}{a}+a\right)$, $BD=\frac{1}{2}\left(\frac{x^2}{a}-a\right)$

$AC=\frac{1}{2}\left(\frac{x^2}{b}+b\right)$, $DC=\frac{1}{2}\left(\frac{x^2}{b}-b\right)$

Hence, $BC=\frac{1}{2}\left(\frac{x^2}{a}+\frac{x^2}{b}-a-b\right)$.

(*B. S. S.* xii, sl. 34)

Mahavira's solution is different. Let mn be factorized in another way as pq. Now m^2-n^2, $2mn$, m^2+n^2 and p^2-q^2, $2pq$, p^2+q^2 are the sides of two right-angled triangles. If AD is chosen so that $AD=2mn=2pq$, the two triangles may be taken as ABD, ACD or as $AB'D$, ACD where $\Delta ABD \equiv \Delta AB'D$ (see Fig. 10). Then the required triangle is ABC or $AB'C$.

Ex. If $AD=12$, we can take $BD=5$, $AB=13$, $DC=9$, $AC=15$. Hence the triangles with sides 13, 14, 15 or 13, 4, 15 have the altitude 12.

This problem and its solutions occur in European mathematics under the names Bachet (1621) and Cunliffe.*

47. *The circum radius of a triangle.* If b and c are two sides, and p the altitude through their point of intersection, the diameter of the circumcircle is bc/p (xii, sl. 27). No proof is given, but since the Hindus were quite familar with the properties of similar triangles, from the Vedic times, Brahmagupta might have proved the result on exactly the same lines as in the present-day books on geometry.

* Dickson, *l. c.*

48. *Construction of cyclic quadrilaterals with rational sides.* The problem of constructing a quadrilateral with rational numbers for its sides is trivial, since the data are insufficient. The problem of constructing cyclic quadrilaterals with rational sides is of interest. Brahmagupta's work in this connection is interesting and noteworthy, though he makes a serious error in not mentioning that the quadrilaterals he obtains are cyclic.*

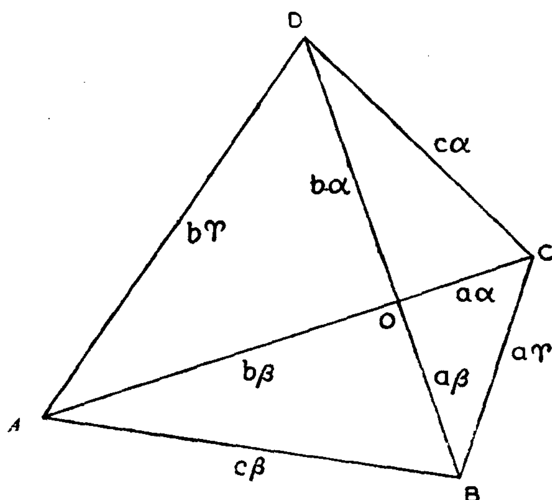


FIG. 11.

Let (a, b, c) and (α, β, γ) be the sides of two right-angled triangles such that $c^2=a^2+b^2$ and $\gamma^2=\alpha^2+\beta^2$. Construct the triangles BOC, COD with sides $(a\alpha, a\beta, a\gamma)$ and $(\alpha a, \alpha b, \alpha c)$ respectively. On the other side of BD, construct the triangles DOA, AOB with sides $(b\alpha, b\beta, b\gamma)$ and $(\beta a, \beta b, \beta c)$ respectively. We thus obtain a quadrilateral $ABCD$ with sides $c\beta$, $a\gamma$, $c\alpha$, $b\gamma$ in order, and having its diagonals at right-angles. It is easy to prove by using §47 that this quadrilateral is cyclic, for the circum radius of $\triangle ABD=\frac{1}{2}\frac{bc\beta\gamma}{b\beta}=\frac{1}{2}c\gamma$, while that of $\triangle ACD$ is $\frac{1}{2}\frac{ca\alpha\gamma}{a\alpha}=\frac{1}{2}c\gamma$. Hence, if S

* In sloka 27, he however mentions चतुर्भुजकोणस्पृशग्वृत्त (circumcircle of the quadrilateral', which means that he is probably considering cyclic quadrilaterals only.

is the circum centre of $\triangle ABD$, the circle with s as centre and $\frac{1}{2}c\gamma$ as radius will pass through A, B, C, D.

Pairs of opposite sides can be conveniently denoted by the notation $c.(\beta, \alpha)$ and $\gamma.(a, b)$. If, however, we construct a quadrilater having these for *adjacent* sides, instead of opposite sides, we get another cyclic quadrilateral having a pair of opposite angles as right angles, and $c\gamma$ as one diagonal.

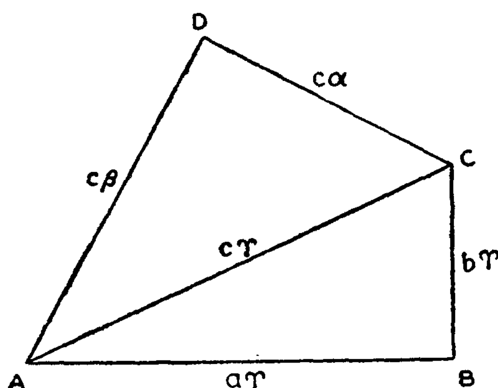


FIG. 12.

Both these constructions give particular solutions only of the general problem. In the first solution, the diagonals are at right angles, while in the second two opposite angles are right angles. The two quadrilaterals with sides $(c\beta, a\gamma, c\alpha, b\gamma)$ and $(c\beta, c\alpha, b\gamma, a\gamma)$ are called *Brahmagupta quadrilaterals*. Taking (3, 4, 5) and (5, 12, 13) for (a, b, c) and (α, β, γ), we obtain the Brahmagupta quadrilaterals with sides (25, 39, 60, 52) and (25, 60, 39, 52). The example is mentioned in Sridhara's *Trisatika* (त्रिशतिका), Ex. 80.

49. There are two other results concerning cyclic quadrilaterals which are known as Brahmagupta's theorems.

(A) *The area of a cyclic quadrilateral with sides a, b, c, d is* $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, *where* $2s=a+b+c+d$.

(B) The diagonals of a cyclic quadrilateral are given by the following rule :

"The sums of the products of the sides about both the diagonals

being divided by each other, multiply the quotients by the sum of the products of the opposite sides ; the square roots of the results are the diagonals in a quadrilateral."

If the sides are a, b, c, d in order, this gives for the diagonals the lengths

$$\sqrt{\left\{\left(\frac{bc+ad}{ab+cd}\right)(ac+bd)\right\}} \text{ and } \sqrt{\left\{\left(\frac{ab+cd}{bc+ad}\right)(ac+bd)\right\}}.$$

Brahmagupta should have got these results from the special quadrilaterals he has constructed. Thus, in Fig. 11

$$AB.BC+AD.DC=(b\alpha+a\beta)c\gamma$$

$$AB.AD+CB.CD=(a\alpha+b\beta)c\gamma$$

$$\therefore \quad \frac{AB.BC+AD.DC}{AB.AD+CB.CD}=\frac{BD}{AC} \tag{1}$$

Also, $AB.CD+AD.CB=(a\alpha+b\beta)(a\beta+b\alpha)=AC.BD.$ (2)*

We thus verify Ptolemy's Theorems. Multiplying (1) and (2), we obtain BD^2. Dividing (2) by (1), we obtain AC^2.

50. There is another type of cyclic quadrilateral given by Brahmagupta, viz. the isosceles trapezium. Once again, Brahmagupta has not realised that the figure is inscribable in a circle.

The problem is stated in the form : To construct a quadrilateral with two opposite sides equal. The answer given is the quadrilateral with sides

$$c, \ \tfrac{1}{2}\left\{\frac{a^2}{k}-k\right\}+b, \quad c, \ \tfrac{1}{2}\left\{\frac{a^2}{k}-k\right\}-b$$

where $c^2=a^2+b^2$ (xii, sl. 36).

The following figure explains the method by which this has been derived.

The sides of ΔADL are c, a, b. A right-angled triangle ALC with a as one side is given by a, $\tfrac{1}{2}\left\{\frac{a^2}{k}-k\right\}$, $\tfrac{1}{2}\left\{\frac{a^2}{k}+k\right\}$ [§45(a)], i.e. $LC=\tfrac{1}{2}\left\{\frac{a^2}{k}-k\right\}$, $AC=\tfrac{1}{2}\left\{\frac{a^2}{k}+k\right\}$. Similarly $DM=\tfrac{1}{2}\left\{\frac{a^2}{k}-k\right\}$. Hence we obtain the lengths of AB and CD as given. When AL is parallel to BM, $ABCD$ is evidently an isosceles trapezium.

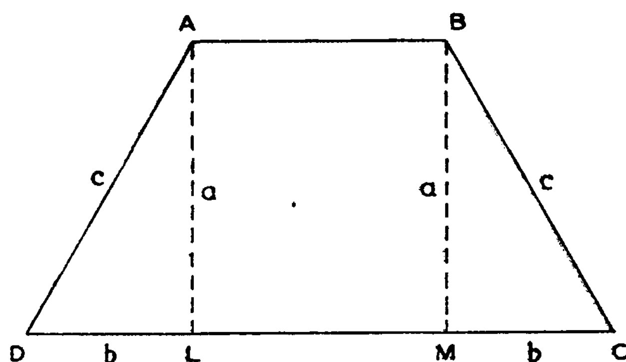


FIG. 13.

Mahavira adopts a somewhat different method (*G. S. S.*, vii, $99\frac{1}{2}$).

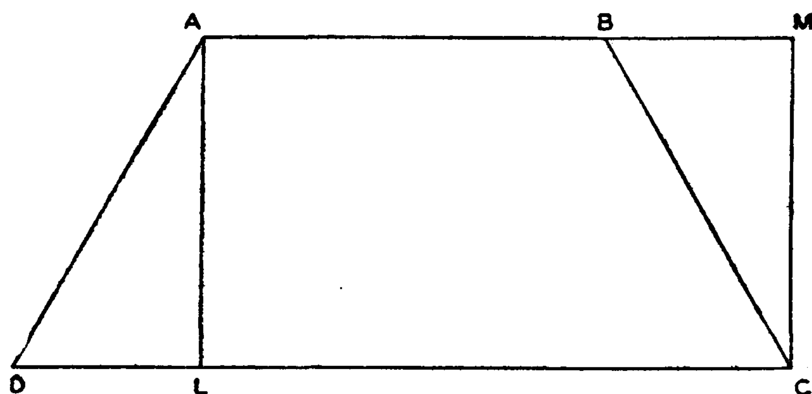


FIG. 14.

Construct the triangle ALD with sides AL, LD, AD respectively equal to $2mn$, m^2-n^2, m^2+n^2. Construct ΔALC with sides $2pq$, p^2-q^2, p^2+q^2 where $pq=mn$. From the side AM of the rectangle $ALCM$, remove $BM=DL$. Then $AD=BC=m^2+n^2$, while

$$AB=(p^2-q^2)-(m^2-n^2),\ CD=(p^2-q^2)+(m^2-n^2).$$

p, q, m, n are to be chosen so that $p^2-q^2>m^2-n^2$.

51. To construct a trapezium with three sides equal.

Solution. The sides are c^2, c^2, c^2, $3a^2-b^2$ where $c^2=a^2+b^2$ (xii, sl. 37).

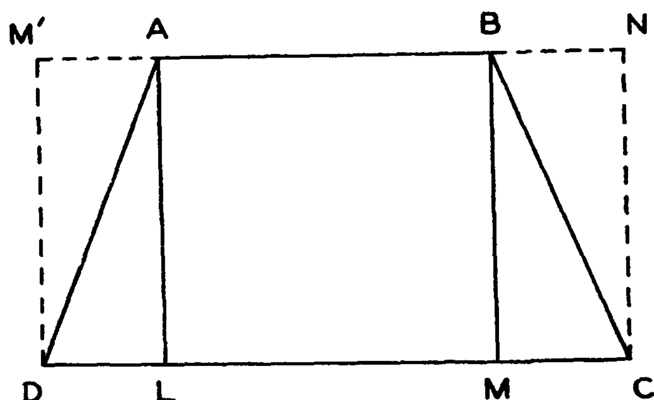


FIG. 15.

Take $$AB=AD=BC=c^2=a^2+b^2$$
$$DL=CM=a^2-b^2$$
$$AL=BM=2ab$$
$$\therefore \quad CD=(a^2+b^2)+2(a^2-b^2)=3a^2-b^2.$$

Taking $c=5$, $b=4$ or 3, we get in particular the solutions (25, 25, 25, 39) and (25, 25, 25, 11) (Prithudakaswami).

Mahavira's solution (G. S. S. vii, $101\frac{1}{2}$) is less simple. He combines two rectangles $ALDM'$ and $ALCN$ (Fig. above), where

$$AL=4mn(m^2-n^2), \quad DL=6m^2n^2-m^4-n^4, \quad CL=8m^2n^2.$$
$$\therefore \quad AB=LC-DL=(m^2+n^2)^2=AD=BC.$$

52. *Interpolation formula.* Brahmagupta wrote another book *Khandakhadyaka* (खंडखाद्यक) in 665 A. D., when he was 67 years old. This is an expository book on astronomy. In chapter IX of this, he discusses a method of obtaining from a given table of sines, the sines of intermediate angles. The following is a table of sines [Brahmagupta takes the radius of the circle as 150].

Angle (in degrees)	Sine	First difference	Second difference
0	0		
		39	
15	39		−3
		36	
30	75		−5
		31	
45	106		−7
		24	
60	130		−9
		15	
75	145		−10
		5	
90	150		

Required sine 57°,* say. According to the rule given by Brahmagupta, the first difference for 57° is

$$\frac{31+24}{2}-\frac{12}{15}\left\{\frac{31-24}{2}\right\}$$

$$\therefore \quad \sin 57^\circ = 106+\frac{12}{15}\left\{\frac{31+24}{2}-\frac{12}{15}\cdot\frac{31-24}{2}\right\}$$

$$=125.76,$$

which compares favourably with 150 sin 57° = 125.80. The rule employed is equivalent to the Newton-Stirling formula

$$f(a+xh)=f(a)+x.\frac{\Delta f(a)+\Delta f(a-h)}{2}+\frac{x^2}{2!}\Delta^2 f(a-h),$$

up to second order differences.

Brahmagupta also gives a rule for interpolation with data at unequal intervals. The meaning of the rule is not clear.†

A new branch of mathematics—Interpolation Theory—was just initiated by Brahmagupta. But neither he nor any succeeding mathematician of India developed the subject further. But Brahmagupta is the world's first mathematician to use second order differences.

* P. C. Sengupta : Brahmagupta on Interpolation. *Bull. Calcutta Math. Soc.* 23(1931), 125–8.

† The illustration given by Sengupta is questionable.

53. Bhaskara was greatly influenced by Brahmagupta's work, and he gave Brahmagupta the title *Ganaka Chakra Chūdāmani* (गणकचक्र चूडामणि, the gem of the circle of mathematicians). Brahmagupta richly deserves the title. In estimating his mathematical work, one should remember that he belonged to the Seventh Century, A. D.

CHAPTER VII

MAHAVIRACHARYA

54. After Brahmagupta, a mathematician of some note is Sridhara who wrote by about 750 A. D. a book on arithmetic by name *Pātīganita Sāra* (पाटीगणित सार), and also a book on algebra. The book on arithmetic contains exactly 300 verses, and is hence known by the name *Trisatika* (त्रिशतिका). The book on algebra has been lost. Bhaskara acknowledges his indebtedness to Sridhara, and quotes Sridhara's statements on one or two occasions.

55. The most celebrated mathematician of the 9th century is Mahaviracharya or briefly Mahavira. He was in the court of the Rashtrakoota king Amoghavarsha Nripatunga (अमोघवर्ष नृपतुंग). The Rashtrakoota dynasty flourished in a part of the present state of Mysore. Mahavira was a Jain by religion. His work *Ganita Sāra Sangraha* (G. S. S. briefly) was written in 850 A. D. The book was discovered some 50 years ago, and has been edited and translated to English by M. Rangacarya of the Madras University, and published by the Government of Madras. The book was widely used in South India, and was translated into Telugu in the 11th century by Pāvutūri Mallana.

Unlike his predecessors, Mahavira was not an astronomer. His work was confined to pure mathematics. The *Ganita Sara Sangraha* is the first text book on arithmetic in the present-day form. The subject matter and the arrangement is practically what one finds in the present-day text books, except for the fact that there is no chapter on decimals. Decimals are not of Indian origin, and the Indians did not know them. According to the tradition of those days, many topics on algebra and geometry have been discussed in the book, and these are of great importance for our discussion.

56. Mahavira's contributions are essentially by way of improving and extending the results of his predecessors, and in providing a large number of what may be called riders. The problems and riders are often long and complicated. These apart, one does not

find in Mahavira's work any profoundly fundamental discoveries. We shall briefly refer to his contributions to arithmetic, algebra and geometry.

57. Arithmetic and Algebra.

(a) "Garland product". The following products resemble a garland i.e. they give the same numbers read from left to right or right to left.

$139 \times 109 = 15151$

$27994681 \times 441 = 12345654321$

$12345679 \times 9 = 111, 111, 111$

$333333666667 \times 33 = 11000011000011$

$14287143 \times 7 = 100010001$

$142857143 \times 7 = 1000000001$

$152207 \times 73 = 11, 111, 111$

$11011011 \times 91 = 1002002001$

(b) Problems on quadratic equations.

(i) चरन्ति कमल षंडे सारसानां चतुर्थो
नवमचरण भागौ सप्तमूलानि चाद्रौ
विकचवकुलमध्ये सप्तनिघ्नाष्टमानाः
कतिकथयसखेत्वं पक्षिणोदक्षसाक्षात्

Out of a certain number of Sārasa birds, one-fourth the number are moving about in lotus plants ; $\frac{1}{9}$th coupled with $\frac{1}{4}$th as well as 7 times the square root of the number move on a hill ; 56 birds remain in Vakula trees. What is the total number of birds ?

If x is the total number of birds, this gives the equation

$$x = \frac{x}{4} + \frac{x}{9} + \frac{x}{4} + 7\sqrt{x} \qquad \therefore \quad x = 576.$$

(ii) $\frac{1}{12}$th part $\times$ $\frac{1}{30}$th part of a pillar is under water, $\frac{1}{20}$th part $\times$ $\frac{3}{16}$th part of the remainder is immersed in mire, leaving a balance of 20 cubits. Dear friend, what is the height of the pillar ?

This gives the equation

$$\left(x - \frac{x^2}{12 \times 30}\right) - \frac{1}{20} \cdot \frac{3}{16}\left(x - \frac{x^2}{360}\right)^2 = 20,$$

where x is the height of the pillar. This gives 4 solutions for x, viz.

240, 120 and two irrational numbers. Mahavira gives only the rational solutions.

Such descriptive problems have been in use in India even prior to Mahavira. Bhaskara gives a large number of elegant and refined problems. Mahavira's problems are generally very involved.

(c) *Values for* a^3. The following formulæ, well-known in present-day elementary algebra, are given by Mahavira.

$$a^3 = a(a+b)(a-b)+b^2(a-b)+b^3$$
$$= a + 3a + 5a + \dots \text{ up to } a \text{ terms}$$
$$= a^2 + (a-1)(1+3+5+7+\dots \text{ up to } a \text{ terms})$$
$$= 3[1.2+2.3+3.4+\dots+(a-1).a]+a.$$

(d) Four different sums of money equal to 40, 30, 20 and 50 respectively are lent out at the same rate of interest for 5, 4, 3 and 6 units of time respectively. The total interest is 34. What is the interest that each amount fetches ?

This is a problem on the application of the following theorem on ratios : If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\dots$, then each ratio is equal to $\frac{a+c+e+\dots}{b+d+f+\dots}$. Mahavira's solution is as follows. Let the rate of interest be r, and x_1, x_2, x_3, x_4 the interests earned by the given amounts.

$$\therefore \quad r=\frac{x_1}{40\times5}=\frac{x_2}{30\times4}=\frac{x_3}{20\times3}=\frac{x_4}{50\times6}=\frac{34}{200+120+60+300}.$$

Hence we can calculate r, x_1, x_2, x_3, x_4.

(e) Three merchants found a purse on the way. One of them said, "If I secure this purse, I shall become twice as rich as both of you with your moneys on hand". Then the second said, "I shall become thrice as rich". The third man said "I shall become 5 times as rich". What is the value of money in the purse, as also the money on hand with each of them ?

Solution. Let x, y, z be the moneys on hand with them, u the money in the purse, a, b, c, the given multipliers, viz. 2, 3, 5. Then

$$u+x=a(y+z)$$
$$u+y=b(z+x)$$
$$u+z=c(x+y)$$

$$\therefore \quad u+x+y+z=(a+1)(y+z)=(b+1)(z+x)=(c+1)(x+y)$$

$$\therefore \quad \frac{(a+1)(b+1)(c+1)}{u+x+y+z}\cdot(y+z)=(b+1)(c+1).$$

Adding this and two similar equations,

$$\frac{2(a+1)(b+1)(c+1)(x+y+z)}{u+x+y+z}=(b+1)(c+1)+(c+1)(a+1)$$
$$+(a+1)(b+1)$$
$$=s, \text{ say}.$$

Hence, we obtain

$$x:y:z=s-2(b+1)(c+1):s-2(c+1)(a+1):s-2(a+1)(b+1)$$
$$=6:18:30 \text{ with the given values of } a, b, c.$$

$$\therefore \quad x:y:z=1:3:5 \text{ and } u:x:y:z=15:1:3:5.$$

(f) We have referred in Chapter III to simple examples on permutations and combinations found in the ancient religious works of the Jains, and in the works of others. Permutations and combinations as a subject of mathematics owes its origin to the ancient Jains. Mahavira is the world's first mathematician to give the general formula

$$_nC_r=\frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots r} \quad \text{(Sl. 218).}$$

(g) *Unit fractions.* Perhaps the most interesting results in Mahavira's work are his methods of obtaining unit fractions for any given fraction. A unit fraction is one whose numerator is unity. The ancient Egyptian mathematician Ahmes (*c.* 1550 B. C.) gave a table expressing the number $\frac{2}{2n+1}$ as the sum of unit fractions, for $n=1$ to 49. His methods were probably empirical. For example

$$\frac{2}{43}=\frac{84}{43\times42}=\frac{43+21+14+6}{43\times42}=\frac{1}{42}+\frac{1}{86}+\frac{1}{129}+\frac{1}{301}$$

which is Ahmes's result. We can also have

$$\frac{2}{43}=\frac{48}{43\times24}=\frac{43+4+1}{43\times24}=\frac{1}{24}+\frac{1}{258}+\frac{1}{1032}$$

$$=\frac{360}{43\times180}=\frac{215+90+45+10}{43\times180}=\frac{1}{36}+\frac{1}{86}+\frac{1}{172}+\frac{1}{774}.$$

In this way, any number can be expressed as the sum of unit fractions in many ways. But Mahavira gives a set of rules which do not savour of these empirical methods.

(i) To express 1 as the sum of n unit fractions. (G.S.S., Sl. 75)

$$1=\frac{1}{2}+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\dots+\frac{1}{3^{n-2}}+\frac{1}{2.3^{n-2}}.$$

Ex. $n=5$; $1=\frac{1}{2}+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{54}.$

(ii) To express 1 as the sum of $2n-1$ (i.e. an odd number) unit fractions. (G. S. S., Sl. 77)

$$1=\frac{1}{2.3.\frac{1}{2}}+\frac{1}{3.4.\frac{1}{2}}+\dots+\frac{1}{(2n-1)2n.\frac{1}{2}}+\frac{1}{2n.\frac{1}{2}}.$$

Ex. $n=4$; $1=\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}+\frac{1}{4}.$

(iii) To express a given unit fraction as the sum of r fractions with numerators $a_1, a_2, \dots, a_r$ respectively (G. S. S., Sl. 78).

We have

$$\frac{1}{n}=\frac{a_1}{n(n+a_1)}+\frac{a_2}{(n+a_1)(n+a_1+a_2)}+\dots$$
$$+\frac{a_{r-1}}{(n+a_1+a_2+\dots+a_{r-2})(n+a_1+\dots+a_{r-1})}$$
$$+\frac{a_r}{a_r(n+a_1+\dots+a_{r-1})}.$$

By taking $a_1=a_2=\dots=a_r=1$, we express in this way $\frac{1}{n}$ as the sum of r unit fractions.

Ex. $n=5, r=4$; $\frac{1}{5}=\frac{1}{30}+\frac{1}{42}+\frac{1}{56}+\frac{1}{8}.$

(iv) To express any fraction as the sum of unit fractions (G. S. S., Sl. 80).

If $\frac{p}{q}$ is the given fraction ($p<q$), determine i so that $\frac{q+i}{p}$ is an integer, say r. Hence $\frac{p}{q}=\frac{1}{r}+\frac{i}{rq}$. Repeat the process for $\frac{i}{rq}$ and, so on. Since $i<p$, the process ends after a finite number of steps.

(v) To express a unit fraction as the sum of two unit fractions (G. S. S., Sl. 85).

First method. $\frac{1}{n}=\frac{1}{pn}+\frac{1}{\frac{pn}{p-1}}$, where p is to be chosen so that $p-1$ divides n.

Second method. $\frac{1}{ab}=\frac{1}{a(a+b)}+\frac{1}{b(a+b)}$, $(b\geqslant 1)$.

(vi) To express any fraction as the sum of two fractions with given numerators (G. S. S., Sl. 87).

We have

$$\frac{m}{n}=\frac{a}{\frac{ap+b}{m}\cdot\frac{n}{p}}+\frac{b}{\frac{ap+b}{m}\cdot n}$$

Here p must be chosen so that p divides n, and m divides $ap+b$. The rule is not of universal application. If we take $p=n$, and if m divides $an+b$, then

$$\frac{m}{n}=\frac{a}{\left(\frac{an+b}{m}\right)}+\frac{b}{\left(\frac{an+b}{m}\right)n}.$$

(vii) Combining (i) and (vi), and subject to the limitations mentioned in (vi), any fraction can be expressed as the sum of $2n$ fractions with given numerators (G. S. S., Sl. 89).

(h) We shall refer to Mahavira's work relating to *Kuttaka* (indeterminate equations) in a subsequent chapter.

58. Geometry. Mahavira repeats Brahmagupta's construction for a cyclic quadrilateral with rational sides, failing to note as Brahmagupta also did, that the figure is cyclic. We have already referred in the previous chapter to Mahavira's solution for some of the subsidiary problems arising out of this problem. Two other subsidiary problems are the following :

(i) To construct a cyclic quadrilateral having a given area A.

Solution. Choose four factors of A^2 such that $A^2=\alpha\beta\gamma\delta$. Then if $s'=\frac{1}{2}(\alpha+\beta+\gamma+\delta)$, the cyclic quadrilateral with sides $s'-\alpha$, $s'-\beta$, $s'-\gamma$, $s'-\delta$ has its area equal to A.

(ii) To construct a cyclic quadrilateral so as to have its circumdiameter equal to D (G. S. S., vii, $221\frac{1}{2}$).

Solution. We have noted that the circumdiameter of the Brahmagupta quadrilateral discussed in §48 is $C\gamma$. Hence if the sides of that quadrilateral are increased in the ratio $\frac{D}{C\gamma}$, we get the required quadrilateral. The sides of the required quadrilateral are therefore

$$\frac{D}{C\gamma}.C\beta,\ \frac{D}{C\gamma}.A\gamma,\ \frac{D}{C\gamma}.C\alpha,\ \frac{D}{C\gamma}.B\gamma.$$

This principle of ratio is used in other problems too by Mahavira and by Bhaskara.

(iii) To find a rectangle whose area is numerically a multiple of the perimeter or the diagonal, or in general a linear combination of the sides and the diagonal (G. S. S., vii, $112\frac{1}{2}$).

If x, y, z are the sides and the diagonal, this means

$$x^2+y^2=z^2 \qquad (1)$$

$$rxy=mx+ny+pz. \qquad (2)$$

Solution. Starting with any solution of $x'^2+y'^2=z'^2$, let $mx'+ny'+pz'=Q$.

The required solutions are obtained by multiplying x', y', z' by the ratio $\frac{Q}{rx'y'}$. Thus,

$$x=\frac{x'Q}{rx'y'}=\frac{Q}{ry'};\ y=\frac{Q}{rx'};\ z=\frac{z'Q}{rx'y'}.$$

It is readily verified that these values satisfy (1) and (2). In particular, if $m=n=0$, $p=r=1$, we obtain the sides and diagonals of a rectangle whose area is numerically equal to its diagonal.

(iv) The perimeter of a rectangle is 1 ; calculate the sides.

Answer : $\frac{m^2-n^2}{2(m^2-n^2+2mn)}$, $\frac{2mn}{2(m^2-n^2+2mn)}$.

(v) Figures with a given area : (G. S. S., vii, 146). The sides of an isosceles trapezium having a given area A are

$$\text{top} = \frac{s^2A - 2mn(m^2 - n^2)}{2mns}$$

$$\text{base} = \frac{1}{s}\left\{\frac{s^2A - 2mn(m^2 - n^2)}{2mn} + 2(m^2 - n^2)\right\}$$

$$\text{side} = \frac{m^2 + n^2}{s}$$

$$\text{altitude} = \frac{2mn}{s},$$

where s is any rational number such that

$$s^2A > 2mn(m^2 - n^2).$$

The sides of a trapezium with three equal sides having a given area A, are given by

$$\text{side} = \frac{1}{2}\left(\frac{A^2}{s^3} + s\right), \ \text{base} = 2s - \frac{1}{2}\left(\frac{A^2}{s^3} + s\right), \ \text{altitude} = \frac{A}{s},$$

where s is arbitrary.

There will be no difficulty in following the rationale of these solutions.

(vi) The following is a harder problem (G. S. S. vii, $131\frac{1}{2}$—33).

To construct two rectangles,

(1) whose perimeters are equal, but the area of one is double that of the other

(2) whose areas are equal but the perimeter of one is double that of the other

(3) the perimeter of one is double that of the other, and the area of the latter is double that of the former.

More generally, these mean the solution of the equations

$$m(x + y) = n(u + v)$$

$$pxy = quv$$

where (x, y), (u, v) are the sides of the two rectangles, and m, n, p, q are given numbers.

Mahavira gives two solutions, but the problem is indeterminate. B. Datta* has worked out the general solution in the form

$$y=r\frac{m^2q^2}{n^2p^2}+t$$

$$x=\frac{rm^2q^2}{tn^2p^2}\left(\frac{rm^2q^2}{n^2p^2}-r\frac{q}{p}+\ \right)$$

$$v=\frac{rmq}{np}$$

$$u=\frac{m}{nt}\left(r\frac{m^2q^2}{n^2p^2}+t\right)\left(r\frac{m^2q^2}{n^2p^2}-r\frac{q}{p}+t\right)$$

where r and t are arbitrary.

Mahavira's work contains many other problems of a similar nature.

59. Mahavira is the only Indian mathematician who has briefly referred to the ellipse (आयतवृत्त, in his terminology). The Greeks studied the conic sections in great detail, but nothing at all on these lines was done in India. Mahavira gives the area of the ellipse as circumference $\times\frac{1}{4}$ semi-minor-axis. This may have been given on the analogy of the area of a circle in the form $\pi d\times\frac{d}{4}$, where d is the diameter. But the formula is quite incorrect.

Mahavira's formula for the perimeter of an ellipse in the form $\sqrt{24b^2+16a^2}$ (as corrected by Rangacarya), where a, b are the semi-axes is worth noting. Putting $b^2=a^2(1-e^2)$, where e is the eccentricity, and writing π instead of $\sqrt{10}$ (the approximate value for π), this may be written in the improved form

$$\text{perimeter}=2\pi a\left(1-\frac{3}{5}e^2\right)^{\frac{1}{2}}.$$

The actual value of the perimeter of an ellipse has to be expressed in terms of elliptic functions. It can be expanded in the form

$$2\pi a\left[1-\frac{e^2}{2^2}-\frac{1^2.3}{2^2.4^2}e^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}e^6-\dots\right].$$

* Bibhutibhusan Dutta : On Mahavira's solution of rational triangles and quadrilaterals. *Bull. Calcutta Math. Soc.* xx (commemoration volume), 1930, p. 287. See also Dickson, *Numbers* II, pp. 486, et seq.

into this new instrument which had been installed in her house, and as fate would have it, a small pearl from the ornament on her nose fell into the sand and got mixed with it. This retarded the movement of the sand, with the result that the marriage was celebrated some time after the time which had been fixed up after careful astrological calculations. So, she lost her husband soon after the marriage. To console her in her life-long grief, Bhaskara taught her arithmetic, and named his work after her.

This story has come down from posterity. It is difficult to say whether there is any basis for this story. But internal evidence is definitely against this story, and it is not unlikely that Bhaskara may have given this as a fancy name for his work. The concluding stanza of the *Leelavati* is a pun on language, and would bear evidence to the statement that the name is one of fancy. There have been other works of earlier dates pertaining to other subjects with the same name, ex. the *Leelavati* of Nemichandra, a book on grammar.

62. The *Leelavati* has been divided into 13 chapters, not by the author, but by later scribes. Commencing with a prayer to Lord Ganēsa, the subject matter treated in the book comprises the following: tables, the number system, the eight operations (addition, subtraction, multiplication division, square, cube, square root and cube root), fractions, zero, rule of three, compound rule of three, mixture, interest, progressions, geometry, mensuration, stacks, saw, piles, shadow problems, *Kuttaka* and permutations. From the mathematical point of view, we need interest ourselves only about the chapters on zero, geometry, mensuration, *Kuttaka* and permutations.

The zero (Shunya or Kha i.e. the sky, शून्य, ख) as it applies to addition, subtraction and multiplication has been treated by Brahmagupta himself, thus:

$$a+0=a, \quad a-0=a, \quad a\times 0=0, \quad \sqrt{0}=0, \text{ etc.}$$

A rough conception of infinity, that any other number when divided by zero, gives infinity, first occurs in Bhaskara's work. As will be remarked later, Bhaskara had some clear notions of the differential calculus. In two examples, he makes statements of the

form $\frac{a\times 0}{b\times 0}=\frac{a}{b}$. Favourably disposed critics write this in the form $\lim_{\epsilon\to 0}\frac{a\times\epsilon}{b\times\epsilon}=\frac{a}{b}$. There is no evidence to the specific use of the infinitesimal by Bhaskara though surely he had worked out some few results of the differential calculus. The symbology of ϵ is of course not present.

He treats zero and infinity in the first chapter of his algebra (*Bijaganita*) also. The statement $\infty\pm K=\infty$ is described in graphic language, thus: At the time of the world's creation, the Infinite and Indestructible Lord Almighty creates crores of beings. At the time of the Great Deluge, all these beings go back to His form, and are immersed in Him. Neither process makes any change in Him.

The chapter dealing with geometry gives a number of problems based on the right-angled triangle, and on similar triangles. These are mostly based on Brahmagupta's work. The area and volume of a sphere are given in the form

$$\text{area of sphere}=4\times\text{area of the circle}$$
$$\text{volume}=\text{area}\times\tfrac{1}{6}\text{ diameter.}$$

The chapter on mensuration gives the prismoidal formula for the volume of the frustum of a pyramid. If the upper and lower faces are rectangles of sides (a, b), (a', b') respectively, then according to Bhaskara, the volume of the frustum is obtained by multiplying the sum of the areas of the upper and lower faces, and the area obtained by multiplying the sums of the sides, by one-sixth the height. In other words,

$$\text{volume of frustum}=\tfrac{1}{6}h[ab+a'b'+(a+a')(b+b')].$$

The formula, confined to square and rectangular pyramids, is quite ancient. It occurs in an Egyptian work dated about 1800 B.C., known now by the name "Moscow Papyrus", and also in ancient Chinese mathematics, as well as in Brahmagupta's work.

Bhaskara gives the following illustration: The top of a well is 12 cubits long and 10 cubits wide, the bottom has half these measurements. If the depth of the well is 7 cubits, how much mud is extracted while digging the well?

Answer: $\frac{1}{6}[(12\times 10)+(6\times 5)+(18\times 15)]\times 7=490$ c. cubits.

63. Bhaskara gives the name *Anka Pāsha* (अंकपाश) to the subject of permutations and combinations. The Jains had called this as *Vikalpa* (विकल्प) or *Bhanga* (भंग). Bhaskara writes down the general values of ${}_nC_r$ and ${}_nP_r$ which had been earlier given by Mahavira, and gives some illustrations. The theorem that the number of permutations of r things, of which k are alike of one kind, l alike of another kind, etc. is $\frac{r!}{k!\,l!\ldots}$ is given, for the first time, by Bhaskara.

A particular problem is given, which is not quite elementary.

A number has 5 digits, and the sum of the numbers forming the digits is 13. If zero is not to be a digit, find the total number of numbers possible.

Thus 91111 is a possible number. Permuting the digits, we obtain other numbers 19111, 11911, 11191, 11119. In this way,

91111, 52222, 13333	give rise to 15	numbers
55111, 22333	„ 20	„
82111, 73111, 64111, 43222, 61222	„ 100	„
72211, 53311, 44221, 44311	„ 120	„
63211, 54211, 53221. 43321	„ 240	„
In all,	495	numbers.

Bhaskara also gives the general result. If n is the number of digits (zero being not a digit), s the sum of the digits, then if $9+n>s$, the total number of possible digits is ${}_{s-1}C_{n-1}$ (Ch. XIII, sl. 274). Bhaskara does not give the proof, which is by no means obvious. It is likely that he may have proved it on the following lines, as the proof is directly based on the above mentioned theorem of Bhaskara on permutations with repetitions.

Write $s=n+m$, so that $m<9$. Hence $m+1$ can be a digit.

Write two groups of numbers $1^1, 1^2, 1^3, \ldots, 1^n$ and $1_1, 1_2, \ldots, 1_m$ where the value of 1^r or 1_r is 1. Fixing up 1^n, say, in the first place, take all possible permutations of the remaining $n+m-1$ symbols (taking all of them), of which $n-1$ are alike of one kind, and m are alike of another kind. Place any of these permutations to the right of 1^n. Bracket each 1^r with the succeeding 1_k's, if any. Finally replace each bracket by its value, taking $1^r=1_k=1$. In

this way, a number is obtained containing n digits, the sum of whose digits is s. To make things clear consider the numerical example given above. Here $n=5$, $s=13$, $m=8$. Then one permutation of the quantities,

$$1^5, 1^4, 1^3, 1^2, 1^1; \ 1_1, 1_2, \ldots\ldots, 1_8$$

is by way of example,

$$1^5\, 1_1\, 1_2\, 1^4\, 1_3\, 1_4\, 1_5\, 1^3\, 1_6\, 1_7\, 1^2\, 1_8\, 1^1.$$

Introducing brackets, as explained, we obtain

$$(1^5 1_1 1_2)(1^4 1_3 1_4 1_5)(1^3 1_6 1_7)(1^2 1_8)(1^1),$$

which corresponds to the number 34321. Similarly another permutation is $1^5\, 1_1\, 1_2\, 1_3\, 1_4\, 1^4\, 1^3\, 1_5\, 1_6\, 1^2\, 1^1\, 1_7\, 1_8$, which gives $(1^5 1_1 1_2 1_3 1_4)$ $(1^4)(1^3 1_5 1_6)(1^2)(1_1 1_7 1_8)$, i.e. the number 51313. In this way, every permutation of the above quantities leads to one definite number satisfying the required property, and every such number corresponds to a definite permutation. Hence the required number of numbers is equal to the number of permutations of $n+m-1$ things taken all together, of which $n-1$ are alike of one kind, and m are alike of a second kind. The required number is therefore $(n+m-1)!/(n-1)!m!$

$$=\frac{(n+m-1)(n+m-2)\ldots\ldots(m+1)}{(n-1)!}$$

$$=\frac{(s-1)(s-2)\ldots\ldots(s-n+1)}{(n-1)!}={}_{s-1}C_{n-1}.$$

The following is a different proof.* Consider the product of the n equal factors

$$(x+x^2+x^3+\ldots+x^9+\ldots)(x+x^2+\ldots+x^9+\ldots)(x+x^2+\ldots+x^9+\ldots)\ldots$$

to n factors. In this product, a term in x^s corresponds to each way of choosing a set of indices, one from each factor, such that their sum is s. Since $s<n+9$, no power of x higher than the 9th can be taken from any factor. Hence the required number in our problem is equal to the coefficient of x^s in the expansion of $(x+x^2+\ldots+x^9+\ldots)^n$, i.e. the coefficient of x^{s-n} in $(1+x+x^2+\ldots)^n$, i.e. in

* Both the proofs are taken from Haran Chandra Banerji's "Colebrooke's Translation of the Lilavati, with notes." Banerji says that the second proof is taken from Mahendra Nath Ray's *Algebra*, pt. II.

$(1-x)^{-n}$. The coefficient is

$$\frac{n(n+1)(n+2)\dots(n+s-n-1)}{(s-n)!}$$

$$=\frac{(s-1)!}{(s-n)!(n-1)!}={}_{s-1}C_{n-1}.$$

64. The *Leelavati* is famous on account of the exceedingly fine recreative problems it contains. It has been the tradition of Indian mathematics from the earliest times to make the subject interesting by introducing such examples, so as to enliven the interests of students who may not be kindly disposed towards mathematics. Bhaskara has studied the examples of this nature given by his predecessors Sridhara, Sripathi, Mahavira and Brahmagupta (or Prithudakaswami), simplified and improved them, and given these in his book. These examples give him an opportunity to display his poetic skill. Most of the examples are adapted from those of his predecessors, but a few may be his own.

We shall describe a few of these problems, confining ourselves only to those dealing with quadratic equations, and right-angled triangles. We give the verse, its meaning and the outline of solution :

(1) बाले मरालकुलमूल दलानिसप्त
तीरे विलास भरमंथरगाण्यपश्यम्
कुर्वंचकेलि कलहं कलहंसयुग्मम्
शेषं जले वदमरालकुल प्रमाणम्

O girl ! out of a group of swans, $\frac{7}{2}$ times the square root of the number are playing on the shore of a tank. The two remaining ones are playing with amorous fight, in the water. What is the total number of swans ?

If x is their number, this gives the equation $\frac{7}{2}\sqrt{x}+2=x$ $\therefore$ $49x=4(x-2)^2$. The roots of the quadratic are $x=16$. and $x=\frac{1}{4}$. The former is the answer to the problem.

This problem may be compared with Mahavira's problem in §57(b).

(2) यातं हंस कुलस्य मूलदशकं मेघागमेमानसं
प्रा‌ड्डीयस्थलपद्मिनी वनमगादष्टांशकोऽम्भस्तदात्
बालेबालमृणालशालिनि जले केलिक्रियालालसं
दृष्टं हंसयुगत्रयं च सकलां यूथस्य संख्यां वद

[According to mythology, the swans go away to Mānasa Sarovar, a sacred lake on the Himalayas, on the advent of the rainy season, and return to the plains after the rains are over].

O ! tender girl, out of the swans in a certain lake, ten times the square root of their number went away to Mānasa Sarovar on the advent of the rains, $\frac{1}{8}$th the number went away to a forest by name Sthala Padmini. Three pairs of swans remained in the tank, engaged in water sports. What is the total number of swans ?

If x is the number, then $10\sqrt{x}+\frac{1}{8}x+6=x \quad \therefore \quad x=144$.

(3) अलिकुलदलमूलं मालतींयातमष्टौ
निखिलनवमभागश्चालिनी भृंगमेकं
निशिपरिमललुब्धं पद्ममध्येनिरुध्दं
प्रतिरणतिरणंतं ब्रूहिकांतेऽलिसंख्यां

Out of a swarm of bees, a number equal to the square root of half their number went to the Mālati flowers ; $\frac{8}{9}$ths of the total number also went to the same place. A male bee enticed by the fragrance of the lotus got into it. But when it was inside it, night fell, the lotus closed, and the bee was caught inside. To its buzz, its consort was replying from outside. What is the number of bees ?

We have the equation $\sqrt{\frac{x}{2}}+\frac{8}{9}x+2=x \quad \therefore \quad x=72$.

In these problems, only one root of the quadratic equation is admissible. A problem where both roots are admissible occurs in Bhaskara's *Bijaganita*.

(4) वनांतराले प्लवगाष्टभागः संवर्गितोवल्गति जातरागः
फूत्कार नादप्रतिनादरुष्टा दृष्टागिरौ द्वादशते कियंतः

In the interior of a forest, a number of apes equal to the square of $\frac{1}{8}$th of their total number are playing with enthusiasm. The remaining 12 apes are on a hill. The echo of their shrieks by the surrounding hills rouses their fury. What is the total number of apes ?

We have $(\frac{x}{8})^2+12=x$. Both solutions $x=16$ and $x=48$ are admissible.

In the following problem, also from *Bijaganita*, both the roots of the quadratic are integers, but only one root is admissible.

(5) यूथात् पंचांशकस्त्र्यूनो वर्गितोगह्वरंगतः
दृष्टः शाखामृगः शाखामारूढो वदतेकति

Out of a party of monkeys, the square of $\frac{1}{5}$th their number diminished by three went into a cave. The one remaining monkey was getting up a tree. What is the total number of monkeys ?

We have $(\frac{x}{5}-3)^2+1=x$, whose two roots are $x=50$, and $x=5$. The latter root is inadmissible.

65. *Problems on right-angled triangles.*

(1) यद्सिम भुवि वेणुर्द्वित्रिपाणिप्रमाणो
गणक पवनवेगादेकदेशे सभग्नः
भुविनृपमित हस्तेष्वंगलग्नं तदग्रं
कथयकतिषु मूलादेषभग्नः करेषु

O ! mathematician, a bamboo standing on level ground is 32 cubits long. It gets broken by a blast of wind and its end touches the ground at a distance of 16 cubits from its foot. At what height was the bamboo broken ?

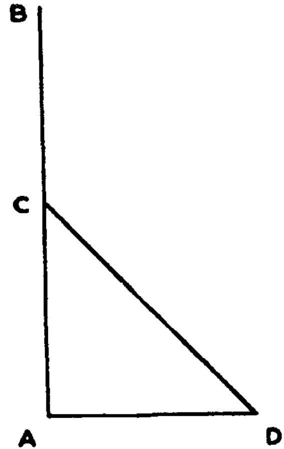


FIG 16.

AB=bamboo=32, $AD=16$. C is the point where the bamboo is broken. If $AC=x$, $BC=CD=32-x$.

$$\therefore \quad x^2+16^2=(32-x)^2 \quad \therefore \quad x=12.$$

(2) अस्तिस्तंभतले बिलं तदुपरि क्रीडा शिखंडी स्थितः
स्तंभेहस्तनवोच्छ्रिते त्रिगुणितस्तंभप्रमाणांतरे
दृष्ट्वाहिं बिलमाव्रजन् तमपतत् तिर्यक् सतस्योपरि
क्षिप्रं ब्रूहि तयोर्बिलात् कतिमितैस्साम्येनगत्योर्युतिः

On a pillar 9 cubits high is perched a peacock. From a distance of 27 cubits, a snake is coming to its hole at the bottom of the pillar.

Seeing the snake, the peacock pounces upon it. If their speeds are equal, tell me quickly at what distance from the hole is the snake caught ?

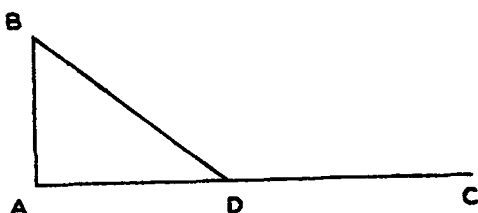


FIG. 17.

AB=pillar=9, AC=27. If $AD=x$, $BD=DC=27-x$.

$$\therefore \quad 9^2+x^2=(27-x)^2 \quad \therefore \quad x=12.$$

(3) चक्र क्रौंचाकुलितसलिले क्वापि दृष्टं तटाके
तोयादूर्ध्वं कमलकलिकाग्रं वितस्तिप्रमाणं
मंदंमंदं चलितमनिलेनाहतं हस्तयुग्मे
तस्मिन्मग्नं गणककथय क्षिप्रमंभः प्रमाणं

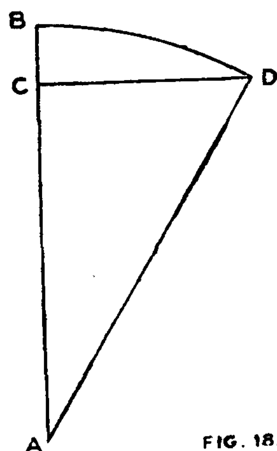


FIG. 18.

In a tank beset with birds, the tip of a lotus is seen at a height of one palm ($=\frac{1}{2}$ cubit). Slowly propelled by the wind, it sinks in the water at a distance of two cubits. What is the depth of water in the tank ? Tell me, quickly, O ! mathematician ?

AB=the lotus, AC=depth of water, CD=water level, $BC=\frac{1}{2}$, $CD=2$. If $AC=x$, $AB=AD=x+\frac{1}{2}$.

$$\therefore \quad (x+\tfrac{1}{2})^2=x^3+4 \quad \therefore \quad x=3\tfrac{3}{4} \text{ cubits.}$$

(4) वृक्षात् हस्तशतोच्छ्रयाच्छतयुगेवापीः कपिः कोऽप्यगात्
उत्तीर्याथपरोद्रुतं श्रुतिपथेनो ड्डीयकिंचिद्द्रुमात्
जातेवं समतातयोर्यदिगता वुड्डायमानं कियत्
विद्वन् चेत् सुपरिश्रमोस्ति गणिते क्षिप्रं तदाचक्ष्वमे

At a distance of 200 cubits from a tree which is 100 cubits high,

is situated a well. Two monkeys are at the top of the tree. One of them climbs down the tree and goes to the well, while the other jumps some distance above the tree, and comes straight to the well. O! learned man, if you are well versed in mathematics, tell me how high did the second monkey jump into the sky, if the distances travelled by the two monkeys are equal.

AB = tree = 100

C = well, $AC = 200$

$BA + AC$ = distance travelled by the first monkey

BD = jump into the sky, by the second

$\therefore \quad BA + AC = BD + DC$

If $BD = x$, $CD = 300 - x$

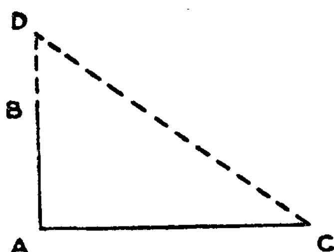


FIG. 19.

$\therefore \quad (300 - x)^2 = (100 + x)^2 + 200^2 \quad \therefore \quad x = 50$ cubits.

We have already remarked that such problems have come down from posterity, and are not necessarily Bhaskara's own. Thus, the problems (2) and (4) above occur with some change of description in the commentary by Prithudaka Swami of Brahmagupta's *B. S. S.* In (2), instead of a peacock and a snake, Prithudakaswami considers a cat and rat. In (4), he considers a hill with two hermits at the top. One hermit climbs down the hill and goes to the pond, while the other by his yogic power jumps into the air, and comes to the pond. Similarly the other two problems (1) and (3) will be found with slight modifications in the ancient Chinese work, *Khyu Chang Suan Shu.* Since the theorem of the right-angled triangle was well known in many countries from early times, it is safe to suggest that such problems might have been independently constructed in either country.

66. Leaving for detailed discussion to a later chapter the subject matters of *Kuttaka* and *Varga Prakriti*, we mention here a few minor topics that will be found in Bhaskara's *Bijaganita.*

(a)(i). Find four different numbers whose sum is equal to the sum of their squares.

Solution : We have $1+2+3+4=10$; $1^2+2^2+3^2+4^2=30$. Hence altering the numbers in the ratio $\frac{10}{30}$, we obtain the required numbers. Thus

$$\tfrac{1}{3}+\tfrac{2}{3}+\tfrac{3}{3}+\tfrac{4}{3}=(\tfrac{1}{3})^2+(\tfrac{2}{3})^2+(\tfrac{3}{3})^2+(\tfrac{4}{3})^2.$$

Similarly, we find four numbers the sum of whose cubes is equal to the sum of their squares. We have $1^3+2^3+3^3+4^3=100$. Hence altering the numbers 1, 2, 3, 4 in the ratio $\frac{30}{100}$, we have

$$(\tfrac{3}{10})^3+(\tfrac{6}{10})^3+(\tfrac{9}{10})^3+(\tfrac{12}{10})^3=(\tfrac{3}{10})^2+(\tfrac{6}{10})^2+(\tfrac{9}{10})^2+(\tfrac{12}{10})^2.$$

(ii) In a right angled triangle, the diagonal is equal to the area (numerically). Find the sides. Similarly, find the sides, if the area is equal to the product of the three sides :

Solution. The area of the triangle with sides 3, 4, 5 is 6. Hence, altering the sides in the proportion $\frac{5}{6}$, we get the solution to the first problem, viz. $\frac{5}{2}$, $\frac{10}{3}$, $\frac{25}{6}$: For the second problem, we multiply by the ratio $\frac{6}{3.4.5}$ or $\frac{1}{10}$. Hence we get the sides $\frac{3}{10}$, $\frac{4}{10}$, $\frac{5}{10}$.

The principle employed in (i) and (ii) gives a useful method in many problems relating to the theory of numbers.

(b) Bhaskara explains the method of extracting the square root of $a+\sqrt{b}$, and also of $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$, where a, b, c, d are rational, pointing out that the latter problem (expressing the square root as the sum of three surds) is not always possible. The example considered by Bhaskara is $10+\sqrt{32}+\sqrt{24}+\sqrt{8}$.

Other problems on surds :

(i) The sides of a triangle are $\sqrt{13}$ and $\sqrt{5}$, the area is 4. Find the base. *Answer* : 4. Bhaskara does not give the other solution $2\sqrt{5}$.

(ii) The sides of a triangle are $\sqrt{10}-\sqrt{5}$ and $\sqrt{6}$, the base is $\sqrt{18}-1$. What is the altitude ? *Answer* : $\sqrt{2}-1$.

(c) A single example of a cubic equation and a single example of a biquadratic are available in the *Bijaganita*. Bhaskara solves the cubic

$$x^3+12x=6x^2+35$$

by writing it as $$x^3-6x^2+12x-8=27$$

$$\therefore\quad (x-2)^3=3^3 \qquad \therefore\quad x-2=3 \quad \therefore\quad x=5.$$

The example is evidently of a special type, and hence no importance can be attached to the solution.

The biquadratic, though again of a special type, is more interesting. To solve

$$x^4-2x^2-400x=9999,$$

Bhaskara adds $4x^2+400x+1$ to both sides, getting

$$(x^2+1)^2=(2x+100)^2$$

$$\therefore \quad x^2+1=2x+100, \quad \text{or} \quad (x-1)^2=100 \quad \therefore \quad x=11$$

Bhaskara's rejecting the solution $x^2+1=-(2x+100)$ is natural, since complex numbers had not been thought of. But he could have given the other solution $x-1=-10$, i.e. $x=-9$.

67. Bhaskara and the Differential Calculus.

Newton (1643—1727) and Leibnitz (1646—1716) are regarded as the founders of the differential and the integral calculus. The notions of the integral calculus had however been understood in a rough way and applied to the determination of areas and volumes by the ancient Greeks. One thus finds the method of summation as envisaged in the integral calculus, from Archimedes to Kepler. Bhaskara determines the area and volume of a sphere by exactly similar methods.* To find the area of the surface of a sphere, Bhaskara gives two methods : In the first method the surface is divided into elementary annuli by drawing a system of parallel circles by taking any point on the surface as centre, and with gradually increasing curvilinear radii. The number of such circles, observes Bhaskara, may be as many as we like, but should conveniently be as many as the number of sines known. The sum of the areas of the elementary annuli gives the area of the sphere. In the second method, the surface is divided into elementary lunes by drawing meridian circles through a pair of diametrically opposite poles on the sphere. Each lune is again divided into a large number of elementary quadrilaterals by drawing circles parallel to the equatorial circle. The sum of the areas of the quadrilaterals gives the area of the lune, and the sum of the areas of all the lunes gives the area of the sphere. To find the volume of the sphere, Bhaskara

* *Siddhanta Siromani—Goladhaya. Bhuvanakosha*, Sl. 57-62. See B. B. Datta : The Hindu Contributions to Mathematics. *Bulletin of the Math. Association*, University of Allahabad, Vols. I and II, 1927-29.

considers a system of pyramids with vertex at the centre and standing on curvilinear areas of unit magnitude on the surface. The sum of the volumes of the pyramids gives the volume of the sphere.

While Bhaskara thus had notions of the integral calculus as much as was known in his time, he deserves special mention as the world's first mathematician who conceived of the differential calculus, and gave the first example of a differential coefficient. To determine accurately the daily motion of a planet, he introduced the *Tātkālika* (instantaneous) method by dividing the day into a large number of small intervals, and comparing the positions of the planet at the end of successive intervals. The *Tatkōlika Gati* (तात्कालिक गति) is essentially the instantaneous motion of the planet. If y and y' are the mean anomalies of the planet at the ends of consecutive intervals, Bhaskara gives

$$\sin y' - \sin y = (y' - y) \cos y$$

which is equivalent to $\delta(\sin y) = \cos y \, \delta y$.

Bhāskara writes this result in the form

बिंबार्धस्य कोटि ज्यागुण स्त्रिज्याहर: फलं दोर्ज्यायोरंतरं

"The product of the cosine of the semidiameter by the element of the radius gives the difference of the two sines."

Bhaskara has gone deeper into the differential calculus and suggests that the differential coefficient vanishes at an extremum value of the function. The idea of Rolle's Theorem is also suggested.

यत्रग्रहस्य परमंफलं तत्रैवगतिफलाभात्रेनभवितव्यं

......यतोवक्रारंभे वक्रत्यागेच गतिः पूर्णंभवति

(*Graha ganita, Spasthadhikara*)

"Where the planet's motion is an extremum, there the fruit of the motion is absent" (i.e. the motion is stationary).

"At the commencement and end of retrograde motion, the apparent motion of the planet vanishes.

These are strong grounds for crediting Bhaskara as the pioneer of the principles of the differential calculus. A rigorous exposition of the subject is of course impossible without the idea of limits, and this idea was formulated in its strict form long after Newton and Leibnitz, while Bhaskara lived 500 years before Newton.

It is a pity that the differential calculus which got such a good beginning at the hands of Bhaskara, was not developed further in India.

A question will be raised whether Bhaskara was at all the pioneer in the subject, or whether rudiments of the calculus were known in India even earlier. According to B. B. Datta (*loc. cit.*), the works of Munjala (मुंजल) (932) and his commentator Prashastidhara (प्रशस्तिधर) (958) show that they were aware of the formula $\delta(\sin\theta) = \cos\theta . \theta$.

68. *Contributions to Trigonometry.* In ancient India, trigonometry was not developed for its own sake as a separate subject, but it was a part and parcel of astronomy. Astronomy was developed to a considerable extent from the earliest times, and the necessary formulæ of plane and spherical trigonometry were worked out as ancillaries to the study of astronomy. We have referred to the table of sines as given by Arya Bhata and in the *Surya Siddhanta*, and the attempts made to understand the construction of this table. Various elementary results relating to the sine and cosine functions will be found in Varāha Mihira's works. Till a systematic and rational study of Indian astronomy is available, a good account of the contributions of the ancient Indians to plane and spherical trigonometry cannot be written. Our study of the history of Indian mathematics has perforce to be incomplete on this score. We shall however refer here to a few results found in Bhaskara's *Siddhanta Siromani*, Part III (*Goladhyaya* or *Jyotpatti*).

(1) The formulæ $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.

चाप्योरिष्टयोः दोर्ज्ये मिथः कोटि ज्यकाहते
त्रिज्या भक्ते तयोरक्यं स्यात् चापकस्य दोर्ज्यका
चापांतरस्य जीवास्यात् तयोरंतरसमिता (Sl. 21—22)

He observes that this rule is useful in finding the sines of other angles.

(2) $\sin\frac{A-B}{2} = \frac{1}{2}[(\sin A + \sin B)^2 + (\cos A - \cos B)^2]^{\frac{1}{2}}$. (Sl. 13)

(3) $\sin 18^\circ = \frac{\sqrt{5}-1}{4}R$; $\sin 36^\circ = \sqrt{\frac{5R^2 - \sqrt{5R^4}}{8}}$,

where R is the radius of the circle. (Sl. 7—9)

69. Bhaskara wrote another work by name *Karana Kutuhala* (करण कुतूहल). The book is not now wellknown.

According to the command of the Mughal emperor Akbar, the *Leelavati* was translated into Persian by his reputed minister Abul Fazl, and the *Bijaganita* was translated into Persian by Utta Ulla Rushudee. These translations were published respectively in 1587, and in 1634 (during Jehangir's reign).

A large number of commentators have written commentaries on the *Leelavati*, Some of these commentaries are well-known and are useful. These include the *Buddhi Vilasani* (बुद्धिविलासिनी) of Ganesa (154), the *Ganitamrita Sara* (गणितामृतसार) of Gangādhara (1420), the *Ganitamrita* (1538) and *Surya Prakāsa* (सूर्यप्रकाश) (1541) of Surya dasa, the commentary of *Vāsana Bhashya* written by Ranganatha (about the beginning of the 17th century) and the *Ganita Kaumudi* (गणित कौमुदी) of Narayana (1356).

Chapter IX

KUTTAKA

70. The solution of the equation $by-ax=c$ for x, y in positive integers, where a, b, c are given integers is called in Hindu mathematics as *Kuttaka*. *Kuttaka* literally means pulverizer, and the name has been given on account of the process of continued division that is adopted for the solution. This equation has been wrongly given the name "Diophantine equation", after the great Greek mathematician Diophantus who lived probably in the latter half of the third century A. D. Diophantus was concerned with the determination of rational solutions of this equation, as also of particular cases of equations of the form $y^2=ax^2+bx+c$. For the linear equation $by-ax=c$, this is a trivial problem, and the name Diophantine equation given to this problem as is understood now is a historical error.

The first mathematician who has dealt with this problem is Arya Bhata of India (499 A. D.). The problem arose to Arya Bhata in the following way. It is required to determine an integer N which when divided by a leaves a remainder r_1, and when divided by b leaves the remainder r_2.

$$\therefore \quad N=ax+r_1=by+r_2$$

or

$$by-ax=c$$

where $c=r_1-r_2$.

The equation

$$ax+c=by \tag{1}$$

where a, b, c are integers, positive or negative, will be called the indeterminate equation of the first degree. Evidently, any factor common to a and b should be a factor of c also, otherwise the equation has no solution. We can then divide by the H. C. F. of a and b, and reduce the equation to the form in which a and b are prime to each other. In what follows, we shall presume that a and b are prime to each other. There is also no loss of generality in taking c to be positive.

71. There is no doubt that Arya Bhata truly discovered the method of solution of this equation, and that this is perhaps the best achievement of his in the field of pure mathematics. But he has given his method in just two stanzas (32 and 33) of his *Arya Bhatiya* in a language which is very difficult to interpret and admits possibly of more than one translation. The paucity of writing material, and the unfortunate habit of writing everything in poetry have contributed immensely to the difficulties of modern commentators. The method would no doubt have been explained personally from master to pupil, and in turn to his pupil and so on, illustrated with numerical examples. In the course of generations, this process may result in the method receiving various modifications at the hands of commentators.

The text of Arya Bhata's rule is as follows :

अधिकाग्र भागहारं छिंद्यादूनाग्र भागहारिणा
शेषपरस्पर भक्तं मतिगुणमग्रांतरे क्षिप्त
अधउपरि गुणितमंत्ययुगूनाग्र च्छेद भाजितेशेषं
अधिकाग्र च्छेदगुणं द्वि च्छेदाग्रमधिकाग्रयुतं

An account of the difficulties of translation and ambiguity of meaning will be found in B. Datta's article in the *Bull. Calcutta Math. Soc.* XXIV (1933), 19–36. In explaining the rule and Arya Bhata's method, we shall follow Datta who bases his interpretation on that of Bhaskara I, who was either a direct pupil of Arya Bhata or belonged to the hierarchy of his pupils.

The translation of Arya Bhata's rule according to the interpretation of Bhaskara I is as follows :

Divide the divisor corresponding to the greater remainder by the divisor corresponding to the smaller remainder. The residue and the divisor corresponding to the smaller remainder being mutually divided, the last residue should be multiplied by an integer at our choice such that the product on being added to (if the number of quotients in the division process is even) or subtracted (if the number of quotients is odd) by the difference of the remainders, will be exactly divisible by the penultimate remainder. Place the quotients of the mutual division succesively one below the other in a column ; below them the optional multiplier, and below

it the quotient just obtained. The penultimate is multiplied by the one just above it, and added to that just below it. Repeating this process, divide the last number obtained by the divisor corresponding to the smaller remainder. Then multiply the residue by the divisor corresponding to the greater remainder, and add the greater remainder. The result will be a number corresponding to the two divisors.

There are some omissions in the rule which have been supplied in the translation.

72. The rule and its rationale can be expressed in modern symbology as follows :

We perform the repeated division as in the H. C. F. process as follows :

$$\begin{array}{l} b)\,a\,(q \\ \underline{bq} \\ r_1)\,b\,(q_1 \\ \quad\underline{r_1q_1} \\ \quad r_2)\,r_1\,(q_2 \\ \qquad\underline{\cdots} \\ \qquad r_3 \\ \qquad\vdots \\ \qquad r_m)\,\underline{r_{m-1}}\,(q_m \\ \qquad\quad r_{m+1} \end{array}$$

If $a<b$, we take $q=0$ and $r_1=a$.

The mutual division can be continued to the finish, or stopped after getting a certain number of quotients. We have

$$a=bq+r_1$$
$$b=r_1q_1+r_2$$
$$r_1=r_2q_2+r_3$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$r_{m-1}=r_mq_m+r_{m+1}$$

Substituting the value of a in (1), (§70) we have

$$by=x(bq+r_1)+c$$
$$\therefore\quad y=qx+y_1 \text{ where } by_1=r_1x+c \tag{1.1}$$

Similarly, from the value of b, and (1.1)

$$y_1(r_1q+r_2)=r_1x+c$$
$$\therefore\quad x=qy+x_1, \text{ where } r_1x_1=r_2y_1-c \tag{1.2}$$

In this way, we have

$y=qx+y_1$	(i)	$by_1=r_1x+c$	(1.1)
$x=q_1y_1+x_1$	(ii)	$r_1x_1=r_2y_1-c$	(1.2)
$y_1=q_2x_1+y_2$	(iii)	$r_2y_2=r_3x_1+c$	(1.3)
$x_1=q_3y_2+x_2$	(iv)	$r_3x_2=r_4y_2-c$	(1.4)
....		....	
....		....	
$y_{n-1}=q_{2n-2}x_{n-1}+y_n$	$(2n-1)$	$r_{2n-2}y_n=r_{2n-1}x_{n-1}+c$	$(1.2n-1)$
$x_{n-1}=q_{2n-1}y_n+x_n$	$(2n)$	$r_{2n-1}x_n=r_{2n}y_n-c$	$(1.2n)$
$y_n=q_{2n}x_n+y_{n+1}$	$(2n+1)$	$r_{2n}y_{n+1}=r_{2n+1}x_n+c$	$(1.\overline{2n+1})$

We may continue the mutual division to the finish or stop at some stage. We consider the two cases separately :

Case I. The mutual division continued to the finish. The number of quotients q_1, q_2, ... (excluding q) may be even or odd.

(i) Suppose the number of quotients is even.

i.e. $$r_{2n}=1, \quad r_{2n+1}=0, \quad q_{2n}=r_{2n-1}$$

$\therefore$ From the equations $(2n)$ and $(1.\overline{2n+1})$,

$$y_n=q_{2n}x_n+c \quad \text{and} \quad y_{n+1}=c.$$

Giving any integral value t for x_n, we obtain y_n. We then obtain x_{n-1} from the equation $(2n)$. Proceeding backwards step by step, we ultimately obtain x and y in positive integers.

(ii) Suppose the number of quotients is odd.

$$\therefore \quad r_{2n-1}=1, \quad r_{2n}=0, \quad q_{2n-1}=r_{2n-2},$$

and $$x_{n-1}=q_{2n-1}y_n-c\ ; \ x_n=-c.$$

Giving any integral value t' for y_n, we obtain x_{n-1}. Proceeding backwards as before, we calculate x and y.

Case II. The mutual division is stopped at some stage.

(i) Number of quotients even. We have

$$r_{2n}y_{n+1}=r_{2n+1}x_n+c.$$

By trial, we choose an integer t for x_n such that

$$y_{n+1}=(tr_{2n+1}+c)/r_{2n}$$

is an integer.

Since the q's and r's are known, we can proceed backwards along the table $(1.1)-(1.\overline{2n+1})$ and get x and y.

(ii) Number of quotients odd. We have now

$$r_{2n-1}x_n = r_{2n}y_n - c.$$

Choose y_n = an integer t' such that $x_n = (r_{2n}t' - c)/r_{2n-1}$ is an integer. We now work the x's and y's backwards as before, and obtain x and y.

It is evident that if $x=\alpha, y=\beta$ be any solution (the least solution, say) of $ax+c=by$ in integers, then $x=\alpha+bm$, $y=\beta+am$ is also a solution, m being any integer. This is therefore the general solution.

73. *Illustrative example* :

$137x+10=60y$

```
60) 137 (2
    120
    ---
     17( 60 (3
         51
         --
         9) 17 (1
             9
             -
            8) 9 (1
               8
               -
               1
```

Form the column

2
3
1
1

The number of quotients omitting the first one is 3. Hence we choose a multiplier such that on multiplication by the last residue viz. 1, and subtracting 10 from the product, the result is divisible by the penultimate remainder 8. We have $1\times18-10=8\times1$.

Hence we form the following table

	2	2	2	2	297
	3	3	3	130	130
	1	1	37	37	
	1	19	19		
the multiplier ←	18	18			
quotient obtained ←	1				

The number 18 and the number above it, in the first column multiplied and added to the number below it gives the last but one number in the second column. Thus $18\times1+1=19$. The same process applied to the second column gives the third, viz, $19\times1+18=37$. Similarly $37\times3+19=130$, $130\times2+37=297$.

Then $x=130, y=297$ are solutions of the given equation. Noting that $297\equiv23 \pmod{137}$, $130\equiv10 \pmod{60}$, we get $x=10, y=23$ as simple solutions. The general solution is $x=10+60m, y=23+137m$.

Suppose now we stop with the remainder 8 in the process of division above. The number of quotients is now even (omitting the first). The multiplier can now be taken as 1, for $8\times1+10=9\times2$. Hence we have

	2	2	2	23
	3	3	10	10
	1	3	3	
multiplier ←	1	1		
quotient ←	2			

We at once get $x=10, y=23$.

74. Arya Bhata's method or rule is sufficiently explained by the two ways given above for solving the above example. Various artifices are employed to simplify the work.

If x_1, y_1 are solutions of $ax+1=by$, then cx_1, cy_1 are solutions of $ax+c=by$. If $cx_1\equiv x'_1 \pmod b$, $cy_1\equiv y'_1 \pmod b$, then x'_1, y'_1 are solutions of $ax+c=by$. x_1, y_1 are called the constant pulverizers (स्थिरकुट्टक). This simplification has been adopted by later writers, including Brahmagupta and Bhaskara II. Thus in the first method in the above example, taking $c=1$ instead of 10, we may take the multiplier as 9, for $9\times1-1=8\times1$. Hence the work becomes

2	2	2	2	153
3	3	3	67	67
1	1	19	19	
1	10	10		
9	9			
1				

67 and 153 are the constant pulverizers.

Then, $67 \times 10 = 670 \equiv 10 \pmod{60}$

$153 \times 10 = 1530 \equiv 23 \bmod (137)$.

We thus get the solutions $x = 10$, $y = 23$ as before.

75. *Mahavira's modification.* Following any of the methods above Mahavira omits the first quotient q. Then the top figure of the last column gives the value of x. Thus, in the above illustration, $x = 130$ or 670 according as the method of §73 or §74 is employed. Since $130 \equiv 10 \pmod{60}$ and $670 \equiv 10 \pmod{60}$. we obtain $x = 10$. The value of y is then obtained by substitution in the given equation.

We take the following example from Mahavira, (Chapter VI, *Ganita Sara Sangraha*).

Solve $63x + 7 = 23y$.

```
23) 63 (2
    46
    ---
    17) 23 (1
        17
        ---
         6) 17 (2
            12
            ---
             5) 6 (1
                5
                --
                1) 5 (4
                   4
                   
                   1
```

Choose the multiplier so that on multiplication by the last remainder, viz. 1 and added to 7, the result is divisible by the penultimate remainder, viz. 1. We can take the multiplier as 1. Hence we have

1	1	1	1	51
2	2	2	38	38
1	1	13	13	
4	12	12		
1	1			
8				

$\therefore \quad x = 51 \equiv 5 \bmod 23$.

$\therefore \quad x = 5$ is the least solution. Substituting in the given equation, we get $y = 14$.

Mahavira has dodged so as to avoid a zero remainder by dividing 5 by 1, so as to get a quotient of 4 and remainder 1. But this is unnecessary. The quotient may be taken as 5 and the remainder 0. We can now take any multiplier, say 0 itself, and hence obtain

1	1	1	1	28
2	2	2	21	21
1	1	7	7	
5	7	7		
0	0			
7				

And now $x = 28 \equiv 5 \bmod 23$, so that $x = 5$ as before.

76. *Bhaskara's modification.* Bhaskara works out the mutual division to the finish, i.e. till the last remainder is 1. Then the first column is taken as the sequence of quotients with c and 0 attached at the end. The rest of the process is as usual. This is in effect equivalent to taking $x_n = 0, y_n = c$ in §72, case I(i) from the equation (2n) of §72, $x_{n-1} = cq_{2n-1} + 0$. Hence the method.

Illustration : $100x + 90 = 63y$.

```
63) 100 (1
     63
    ---
    37) 63 (1
        37
        --
        26) 37 (1
            26
            --
            11) 26 (2
                22
                --
                 4) 11 (2
                     8
                    --
                     3) 4 (1
                        3
                        -
                        1
```

1	1	1	1	1	1	2430
1	1	1	1	1	1530	1530
1	1	1	1	900	900	
2	2	2	630	630		
2	2	270	270			
1	90	90				
90	90					
0						

$\therefore \quad y=2430\equiv 30 \bmod 100$

$x=1530\equiv 18 \bmod 63$

$\therefore \quad x=18, y=30$, or in general $x=63t+18, y=100t+30$.

If however the number of quotients is odd, this method gives the solution of the equation $ax-c=by$. We could adopt the method by taking $-c$ in place of c. Otherwise, if α and β are the solutions obtained for the equation $ax-c=by$, then

$$x=(b-\alpha)+bt, \quad y=(a-\beta)+at$$

are the solutions of $ax+c=by$, as may be verified.

Illustration. $60x+16=13y$.

We write down the table only

4	4	4	4	4	368
1	1	1	1	80	80
1	1	1	48	48	
1	1	32	32		
1	16	16			
16	16				
0					

$368\equiv 8 \pmod{60}$, $80\equiv 2 \pmod{13}$.

$x=2, y=8$ are solutions of $60x-16=13y$

$x=(13-2)+13t$, $y=(60-8)+60t$ are the solutions of the given equation.

77. Bhaskara explains that the numerical work can be reduced if the numbers (a, c) or (b, c) have a common factor.

(i) Suppose $a=ka'$, $c=kc'$ where k is an integer. The equation reduces to $a'x+c'=by'$, where $y'=y/k$. Hence if (x, y') are solutions of the reduced equation, (x, ky') are solutions of the original equation.

(ii) Suppose $c=kc'$, $b=kb'$. Then proceeding similarly, if (x', y) are solutions of $ax'+c'=b'y$, then (kx', y) are solutions of the original equation.

(iii) It may be possible to combine cases (i) and (ii).

Let $\qquad a=ka', c=kc'$

$$c'=lc'', \ b=lb''$$

Then the equation becomes

$$ka'x + klc'' = lb''y$$

or $$a'X + c'' = b''y, \quad \text{where } X = \frac{x}{l}, \quad Y = \frac{y}{k}.$$

Hence, if (X, Y) are solutions of the last equation, (lX, kY) are solutions of the given equation.

78. If $x = \alpha$, $y = \beta$ be solutions of $ax + c = by$, we have in the examples in the preceding sections, obtained simpler solutions, (α', β') say, by writing $\alpha \equiv \alpha'$ mod b, $\beta \equiv \beta'$ mod a. This is correct only when $\frac{\alpha - \alpha'}{b} = \frac{\beta - \beta'}{a} =$ the same integer. This will be evident from the form of the general solutions.

For example, consider the equation $5x + 23 = 3y$. The *Kuttaka* method gives $x = 23$, $y = 46$. Now $23 = 3.7 + 2$, $46 = 5.9 + 1$. But $x = 2, y = 1$ are not solutions. We have to write $46 = 5.7 + 11$. Hence $x = 2, y = 11$ are solutions.

79. Another artifice due to Bhaskara is when $a > b > c$, and $a = kb + a'$.

Let (x_1, y_1) be solutions of $a'x + c = by$. Then $(x_1, y_1 + kx_1)$ are solutions of $ax + c = by$.

80. If $x = \alpha$, $y = \beta$ are solutions of $ax + c = by$, then $x = \alpha - b$, $y = a - \beta$ are solutions of $ax + by + c = 0$ (Bhaskara).

Example. The smallest solutions of $13y = 60x + 3$ are $x = 11, y = 51$. Therefore $x = -2, y = 9$ are solutions of $60x + 13y + 3 = 0$.

81. *Simultaneous equations.* The *Kuttaka* problem was first conceived by Arya Bhata as the problem of determining a number N which when divided by a leaves r_1 as remainder, and when divided by b leaves r_2 as remainder. Generalizing the problem, we shall require a number N which on being divided by $a_1, a_2, \ldots, a_n$ leaves respectively the remainders $r_1, r_2, \ldots, r_n$. Hence $N = a_1x_1 + r_1 = a_2x_2 + r_2 = a_3x_3 + r_3 = \ldots = c_nx_n + r_n$. These equations can also be expressed in the form $by_1 = a_1x + c_1$, $by_2 = a_2x + c_2$, $by_3 = a_3x + c_3$, etc., for we have

$$a_2x_2 = a_1x_1 + (r_1 - r_2) = a_1x_1 + c_1 \text{ say}$$

$$\therefore \quad a_2x_3 = \left\{\frac{a_1x_1 + (r_1 - r_3)}{a_3}\right\}a_2 = a_1'x_1 + c',$$

etc. The coefficients can be all made integral by multiplying by the *L. C. M.* of a_3, a_4, etc.

The method of solution is simple. Starting from

$$N=a_1x_1+r_1=a_2x_2+r_2,$$

we obtain as previously explained the smallest value α for x_1. Hence the smallest value for N is $a_1\alpha+r_1$, and the general value is

$$N=a_1(a_2t+\alpha)+r_1$$
$$=a_1a_2t+(a_1\alpha+r_1)$$

The *smallest value for N is thus the remainder corresponding to the division of N by the product of the two divisors a_1 and a_2.* The words द्विच्छेदाग्रं in Arya Bhata's rule have been considered to be capable of this interpretation.

We now take

$$N=a_1a_2t+(a_1\alpha+r_1)=a_3x_3+r_3,$$

and solve as before. Proceeding in this way successively, we obtain a value of N satisfying all the given relations.

Illustrations. (1) To find the number which when divided by 8 leaves 5 as remainder, divided by 9 leaves 4 as remainder, and divided by 7 leaves 1 as remainder (Bhaskara I).

The least value of N satisfying

$$N=8x+5=9y+4$$

is easily obtained as 13. Hence by Arya Bhata's rule, 13 is the remainder when N is divided by 72.

$$\therefore\quad N=72t+13=7z+1$$

By the Kuttaka method, we obtain $t=1+7u$. Hence the general value of N is $85+504u$. The least value is 85.

(2) To find N which on division by 6, 5, 4, 3 respectively leaves the remainders 5, 4, 3, 2 (Bhaskara II).

$$\therefore\quad N=6x+5=5y+4=4z+3=3w+2.$$

From the equation $4z+3=3w+2$, we obtain by Kuttaka,

$$w=3+4t,\ z=2+3t$$

This gives $5y+4=12t+11$.

By Kuttaka, $y=11+12u,\ t=4+5u$.

Hence $6x+5=60u+59$, or $x=10u+9$

which is integral, and no further work is required

$$\therefore \quad N=60u+59$$

Or, we may proceed from the other end.

$$6x+5=5y+4 \text{ gives } x=4+5t, \; y=5+6t.$$

$$\therefore \quad N=30t+29=4z+3,$$

giving $\quad t=2u+1 \qquad \therefore \quad N=60u+59.$

This happens to satisfy the last condition, viz. giving the remainder 2 on division by 3. Hence no further work is required, and $N=60u+59$ is the complete solution.

(3) Suppose at a certain time from the beginning of Kalpa, the sun, moon etc., have travelled for the following number of days, after completing full revolutions :

Sun	Moon	Mars	Mercury	Jupiter	Saturn
1000	41	315	1000	1000	1000

Given that the sun completes 3 revolutions in 1096 days, moon 5 revolutions in 137 days, Mars 1 in 685 days, Mercury 13 in 1096 days, Jupiter 3 in 10960 days, Saturn 1 in 10960 days, find the number of days elapsed since the beginning of the Kalpa [Brahmagupta, XVIII, sl. 7—8].

Since $137\times8=1096$, and $1000=137\times7+41$, any number x satisfying $x-1000\equiv0 \pmod{1096}$ also satisfies $x-41\equiv0 \pmod{137}$. Hence we omit the moon from our discussion. Considering the Sun and Mars, we have

$$x-1000=1096y$$

$$x-315=685z$$

This gives $\quad 1096y+685=685z$

Dividing by $\quad 137, \quad 8y=5(z-1).$

$y=5$ is the smallest solution. Hence,

$$x=1000+5480=6480.$$

This number satisfies the required condition for Mercury. Taking $y=10$, the value of x, viz. $10960+1000$ will satisfy the conditions for all the planets. The number of days is thus 11960.

(4) Find the least number which when multiplied by 8 and divided by 29 gives the remainder 4, and when multiplied by 17 and divided by 45 gives the remainder 7 (Parameshwara).

We have $\qquad 8x=29y+4, \quad 17x=45z+7$

By Kuttaka, the solution for x in the first equation is $15+29t$, and in the second equation it is $11+45u$.

$$\therefore \quad x=15+29t=11+45u$$

Solving, $\qquad 29t+4=45$ by Kuttaka,

$$t=34+45v$$

$$\therefore \quad x=15+29(34+45v)$$

The least value of x is 1001.

82. These problems involving simultaneous equations have been called संश्लिष्टकुट्टक (the conjunct pulverizer). A slightly more general case is the set of equations

$$b_1y_1=a_1x+c_1,\ b_2y_2=a_2x+c_2,\ b_3y_3=a_3x+c_3, \text{ etc.}$$

Let α_1 be the least value of x satisfying the first equation, so that $b_1m+\alpha_1$ is the general value of x satisfying this equation. Similarly let the general value of x satisfying the second equation be $b_2n+\alpha_2$.

$$\therefore \quad b_2n+\alpha_2=b_1m+\alpha_1$$

Solving for m and n, we obtain the general value of $b_1m+\alpha_1$ i.e. of x satisfying both the equations. We next equate this to the general value of x satisfying the third equation, and repeat the process.

A somewhat different method given by Bhaskara in four palm-leaf manuscript copies of the *Lilavati* has been explained by S. K. Ganguly.*

As before, let α_1 be the least value and $b_1m+\alpha_1$ the general value of x satisfying the first equation $b_1y_1=a_1x+c_1$.

$$\therefore \quad b_2y_2=a_2b_1m+(a_2\alpha_1+c_2).$$

If $m=\mu$ is a particular solution of this, $m=b_2n+\mu$ is the general solution.

$$\therefore \quad x=b_1(b_2n+\mu)+\alpha_1=b_1b_2n+\alpha_2,$$

where $\alpha_2=b_1\mu+\alpha_1$.

* *Bull. Calcutta Math. Soc.* 17(1926), 89-98. Two of the copies in Telugu script are in the Oriental Libraries of Mysore and Madras.

Substituting this value in the third equation, we find the least value of n and hence a value of x satisfying the three equations. And so on.

Illustration. Five heaps of fruits together with two fruits were divided equally among 9 travellers ; 6 heaps together with 4 fruits were divided amongst 8 ; 4 heaps together with 1 fruit were divided amongst 7. Find the number of fruits in each heap. [*Mahavira, Ganita Sara Sangraha*, vi, $129\frac{1}{2}$].

We have the equations

$$9y_1=5x+2,\ 8y_2=6x+4,\ 7y_3=4x+1.$$

We readily get $x=5+9t$ as the general solution of the first equation. We may now follow either of the methods explained above. We outline the work.

(i) *Fist method.* Substituting for x in the second equation, we get $27t+17=4y_2$. Solving, we have $t=1+4m$. The third equation similarly gives $7y_3=36t+21=144m+57$.

Hence, finally $m=7n-2, y_3=144n-33$.

$$\therefore \quad 4x+1=1008n-231$$

or

$$x=252n-58.$$

The smallest value of x is therefore $252-58=194$.

(ii) *Second method.* From the equation $6x+4=8y_2$,

we get

$$x=2+4u$$

$$\therefore \quad 2+4u=59t.$$

This is satisfied by $t=1+4v$

$$\therefore \quad x=5+9(1+4v)=36v+14.$$

The third equation $7y_3=4x+1$ is satisfied by

$$x=5+7p$$

$$\therefore \quad 36v+14=5+7p.$$

This is satisfied by $v=5+7n$.

$\therefore$ Finally,

$$x=36(5+7n)+14=252n+194$$

$\therefore$ The smallest value of x is 194.

We write down the following problems, whose solution will be on the same lines.

(1) The dividends are the 16 numbers beginning with 35 and increasing successively by 3 ; the divisors are 32 and others increasing successively by 2 ; the remainders are 1 and others increasing successively by 2. What is the unknown multiplier ? (*Ganita Sara Sangraha*, vi, 138½).

We have $32y_1 = 35x + 1$, $34y_2 = 38x + 4$,

$36y_3 = 41x + 7$, etc.—16 equations.

(2) Tell me that number which multiplied by 7 and divided by 62 leaves the remainder 3 ; that number again when multiplied by 6 and divided by 101 leaves the remainder 5 ; and when multiplied by 8 and divided by 17 leaves the remainder 9 [Bhaskara, in the palm-leaf copies referred to]

83. A few examples occur in the works of Brahmagupta. Bhaskara and others, of indeterminate equations involving more than two unknowns. The equations are reduced to equations in two unknowns only, arbitrarily assuming the values of the others. There is scarcely anything of special interest, in these problems.

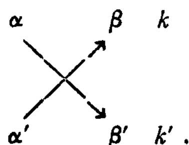
CHAPTER X

VARGA PRAKRITI

or the equation of the multiplied square

84. The word *Prakriti* here means coefficient, and refers to the coefficient N in the indeterminate equation $Nx^2+1=y^2$, where N is a positive integer. Due to a mistake on the part of Euler, this equation was given the name Pell's equation. Pell just referred to this problem in a book on algebra that he wrote. The equation was nearly solved by Brahmagupta (628), and the solution was improved by Bhaskara (1150), and it is therefore fitting that this equation be called the *Brahmagupta-Bhaskara* equation. The complete theory underlying the solution was expounded by Lagrange in 1767, and rests on the theory of econtinud fractions. The Indian method while involving an element of trial-process makes no reference at all to continued fractions.

85. *Brahmagupta's Solution.* For conveniently chosen values of k and k', let (α, β) and (α', β') be a set of solutions of $Nx^2+k=y^2$ and $Nx^2+k'=y^2$. Then it is readily verified that $x=\alpha\beta'\pm\alpha'\beta$, $y=\beta\beta'\pm N\alpha\alpha'$ are solutions of $Nx^2+kk'=y^2$. This is known as *Brahmagupta's Lemma* [*Brahma Sphuta Siddhanta**, xviii, 64-65], and was rediscovered by Euler in 1764. It will be called the *principle of composition*, or *Samāsa* (समास). The values of x and y are readily remembered by writing



* We shall abbreviate this hereafter as B. S. S.

In particular, putting $k=k'$, it follows that if $N\alpha^2+k=\beta^2$, then $x=2\alpha\beta, y=\beta^2+N\alpha^2$ is a solution of $Nx^2+k^2=y^2$. Hence we have

$$N\left(\frac{2\alpha\beta}{k}\right)^2+1=\left(\frac{\beta^2+N\alpha^2}{k}\right)^2.$$

$x=\dfrac{2\alpha\beta}{k}$, $y=\dfrac{\beta^2+N\alpha^2}{k}$ is therefore a solution of the equation $Nx^2+1=y^2$, and we shall have got a set of integral solutions, if only the above values are integers.

(i) Suppose $k=\pm 1$; then the above are evidently integers.

(ii) Suppose $k=\pm 2$. We have then $x=\alpha\beta$ (taking the positive sign), $y=\frac{1}{2}(\beta^2+\beta^2-2)=\beta^2-1$. x and y are thus integers.

(iii) Suppose $k=4$. Then $x=\frac{1}{2}\alpha\beta$, $y=\frac{1}{2}(\beta^2-2)$. If α is even, then since $N\alpha^2+4=\beta^2$, β is also even. Hence we obtain a pair of integral solutions. If α is odd, performing the Samāsa operation between

$$\tfrac{1}{2}\alpha\beta \qquad \tfrac{1}{2}(\beta^2-2) \qquad 1$$

and

$$\tfrac{1}{2}\alpha \qquad \tfrac{1}{2}\beta \qquad 1,$$

Brahmagupta obtains

$$x=\tfrac{1}{2}\alpha(\beta^2-1), \quad y=\tfrac{1}{2}\beta(\beta^2-3)$$

which are both integers when β is odd. When β is even, the earlier values of x and y are integral.

(iv) Suppose $k=-4$. Then, by the above process

$$N(\tfrac{1}{2}\alpha\beta)^2+1=\{\tfrac{1}{2}(\beta^2+2)\}^2.$$

Applying Samāsa to $\frac{1}{2}\alpha\beta$ $\frac{1}{2}(\beta^2+2)$ 1 with itself, and eliminating N by using the above equation, we obtain

$$x=\tfrac{1}{2}\alpha\beta(\beta^2+2), \quad y=\tfrac{1}{2}(\beta^4+4\beta^2+2),$$

as solutions of $Nx^2+1=y^2$. Applying Samāsa again between

$$\tfrac{1}{2}\alpha\beta \qquad \tfrac{1}{2}(\beta^2+2) \qquad 1$$

and

$$\tfrac{1}{2}\alpha\beta(\beta^2+2) \qquad \tfrac{1}{2}(\beta^4+4\beta^2+2) \qquad 1,$$

we obtain

$$x=\tfrac{1}{2}\alpha\beta(\beta^2+1)(\beta^2+3)$$

$$y=(\beta^2+2)\{\tfrac{1}{2}(\beta^2+1)(\beta^2+3)-1\}.$$

These are integers whether β is odd or even.

We have thus obtained a solution in integers for the equation $Nx^2+1=y^2$, provided that we can have a solution (α, β) for the equation $Nx^2+k=y^2$, when $k=\pm 1$, ± 2 or ± 4. Having got one solution, an infinite number of solutions can be obtained by repeated applications of the Samāsa process. This is Brahmagupta's method —a remarkable feat when we realise that this was done in 628 A. D. and without the use of continued fractions.

86. Brahmagupta's method would have been perfect if only a method other than that of trial be available for solving the equation $Nx^2+k=y^2$, when $k=\pm 1$, ± 2, or ± 4. Bhaskara provides such a method, whieh he calls *Chakra-vāla* (चक्रवाल) or the *cyclic process*. This is as follows:

We can find a and b such that $Na^2+k=b^2$, for *any suitable k*. We have also*, $N.1^2+(m^2-N)=m^2$. Applying Samāsa between $(a\ b\ k)$ and $(1\ m\ m^2-N)$, we readily obtain

$$N\left(\frac{am+b}{k}\right)^2+\frac{m^2-N}{k}=\left(\frac{bm+Na}{k}\right)^2.$$

By the *kuttaka* method (Chap. IX), we choose m so that $am+b$ is divisible by k, choosing m conveniently so that m^2-N is numerically small.

Writing

$$\frac{am+b}{k}=a_1, \quad \frac{m^2-N}{k}=k_1, \quad \frac{bm+Na}{k}=b_1,$$

we have

Bhaskara's Theorem 1. When a_1 is an integer, b_1 and k_1 are also integers.

We have now $Na_1^2+k_1=b_1^2$, by the above. Using a_1, b_1, k_1 instead of a, b, k, we now repeat the above process, obtaining another set of integers a_2, b_2, k_2 such that $Na_2^2+k_2=b_2^2$.

We repeat the process. We have now

Bhaskara's Theorem 2. After a finite number of repetitions, we obtain $N\alpha^2+l=\beta^2$, where $l=\pm 1$, ± 2 or ± 4.

Starting from $Na^2+k=b^2$, where k is any convenient integer whatever, we can thus arrive at a solution of $Nx^2+l=y^2$, where $l=\pm 1$, ± 2 or ± 4. Having got this solution, Brahmagupta's method will lead to an integral solution of the original equation $Nx^2+1=y^2$.

* Bhaskara evidently has taken this from Sripathi (1039), see §90.

87. Bhaskara has not given the proofs of his theorems. The Indian mathematicians never paid sufficient attention towards proofs. The proof of Theorem 1 is simple, though not obvious.

The proof of Theorem 2 is not at all simple. Bhaskara may not have proved it at all, but may have verified the truth of it in concrete instances.

Proof of Theorem 1.

(A) By Datta and Singh.*

We have

$$a_1k=an+b,\ b_1k=bn+Na$$

$$\therefore\quad k(a_1n-b_1)=a(n^2-N)$$

We can conveniently take k prime to a. Hence a_1n-b_1 is divisible by a.

$$\therefore\quad \frac{a_1n-b_1}{a}=\frac{n^2-N}{k}=k_1.$$

k_1 is therefore an integer.

Next,

$$b_1=\frac{bn+Na}{k}=\frac{n(a_1k-an)+Na}{k}$$

$$=a_1n-\frac{(n^2-N)a}{k}$$

$$=a_1n-ak_1=\text{an integer.}$$

(B) By Hankel.

Since $\quad a_1k=an+b,\ k=b^2-Na^2,$

we get $\quad a_1(b^2-Na^2)=an+b$

$$\therefore\quad b(a_1b-1)=a(n+Naa_1)$$

Since a and b may be taken to be prime to each other, a must divide a_1b-1.

Eliminating n between $a_1k=an+b$ and $b_1k=bn+Na$, we get $a_1b-ab_1=1$.

$$\therefore\quad b_1=\frac{a_1b-1}{a}=\text{an integer.}$$

* *History of Hindu Mathematics*, pt. II, p. 172.

Next,

$$n^2-N=\frac{(a_1k-b)^2-Na^2}{a^2}=\frac{k(a_1^2k-2ba_1+1)}{a^2}.$$

Since k is assumed to be prime to a, a^2 divides $a_1^2k-2ba_1+1$.

$$\therefore \quad k_1=\frac{n^2-N}{k}=\frac{a_1^2k-2ba_1+1}{a^2}=\text{an integer.}$$

A very lengthy proof* of Bhaskara's second theorem has been given by A. A. Krishnaswami Ayyangar, who remarks that Bhaskara's Theorem is not quite correct as it stands, but that after a finite number of applications of the Chakravāla process, one obtains the value $l=1$ always, and not necessarily the other values $l=-1$, ± 2, ± 4. If however, one of these values is obtained, the solution of the given equation can then be completed by Brahmagupta's method.

88. *Illustrations* : (1) $61x^2+1=y^2$.

This is a remarkable example of great historical interest. It is one of the examples worked by Bhaskara in his *Bijaganita*. This was one of the problems proposed by Fermat in 1657 as a challenge problem to Frenicle. The smallest solution for x and y runs to 9 and 10 digits respectively. There is no evidence to prove that Fermat knew about Bhaskara-Brahmagupta work. Colebrooke's English translation of Bhaskara's *Bijaganita* was published only in 1817. It is a remarkable example which will illustrate the difference between the Indian method, after Brahmagupta and Bhaskara, and the European method, after Lagrange. Lagrange's method requires the twenty-second convergent of the simple continued fraction for $\sqrt{61}$, while the Brahmagupta-Bhaskara method gives the solution in a few easy steps. We write down both the methods for comparison.

Brahmagupta-Bhaskara method : We have $61.1^2+3=8^2$, so that $a=1$, $k=3$, $b=8$. We choose m so that $\frac{m+8}{3}$ is an integer, such that $\frac{m^2-61}{3}$ is small. We take $m=7$. Hence $a_1=5$, $k_1=-4$, $b_1=39$.

$\therefore$ $61.5^2-4=39^2$. We have thus obtained at one step an equation $Nx^2+l=y^2$, where $l=-4$.

* *Jour. Ind. Math. Soc.* Vol. 18 (First Series), Second Part, 232-45.

Performing Samāsa between $\frac{5}{2}\ \frac{39}{2}\ -1$ and $\frac{5}{2}\ \frac{39}{2}\ -1$,
we obtain $\frac{195}{2}\ \frac{1523}{2}\ 1$.

We perform Samāsa between $\frac{195}{2}\ \frac{1523}{2}\ 1$ and $\frac{5}{2}\ \frac{39}{2}\ -1$,
obtaining $\alpha=3805,\ \beta=29718,\ l'=-1$.

Finally performing Samāsa on this with itself, we obtain

$$x=226{,}153{,}980$$

$$y=1{,}766{,}319{,}049$$

as the solution (smallest) of $61x^2+1=y^2$.

Lagrange's method. We have

$$\sqrt{61}=7+\underset{*}{\frac{1}{1+}}\frac{1}{4+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{4+}\frac{1}{1+}\underset{*}{\frac{1}{14}}.$$

The number of quotients in the periodic cycle is 11. Hence the smallest solution for (x, y) is (q_{22}, p_{22}). See Barnard and Child: *Higher Algebra*, Chap. XXXIII for the theory underlying this, as also for the short cuts in the calculation of p_{22} and q_{22}.

The method of continued fractions is however powerful inasmuch as it gives a formula for the general solution. If p, q be the least positive integral solution of $Nx^2+1=y^2$, the general solution is given by equating the rational and irrational parts in the equation

$$\sqrt{N}x+y=(\sqrt{N}q+p)^t, \text{ where } t=1, 2, 3, \ldots$$

(2) $67x^2+1=y^2$.

This example illustrates the *Chakravala* process better than the previous one, wherein the prescribed form $(l=-4)$ is obtained at a single step.

We have $67\cdot1^2-3=8^2\quad\therefore\quad a=1,\ b=8,\ k=-3$.

$\frac{m+8}{-3}$ is an integer if $m=7$, as before. This gives $a_1=-5, k_1=6, b_1=41$

$\therefore\quad 67.5^2+6=41^2$.

We now choose p conveniently, so that $\frac{5p+41}{6}$ is an integer. We take $p=5$, so that $a_2=\frac{pa_1+b_1}{k_1}=-11$,

$$k_2=\frac{p^2-67}{k_1}=-7,\qquad b_2=\frac{b_1p+67a_1}{k_1}=-90.$$

$$\therefore\quad 67\cdot11^2-7=90^2.$$

We next choose q so that $\frac{-11q-90}{-7}$ is an integer.

We take $q=9$.

$$\therefore \quad a_3=27,\ k_3=-2,\ b_3=221.$$

We have thus reached one of the prescribed values of k. The solution is now completed. Samāsa between

	27	221	−2
and	27	221	−2
gives	11934	97684	4

Dividing by 2 (i.e. $\sqrt{4}$), we get the required solution

$$x=5967, \qquad y=48842.$$

(3) $8x^2+1=y^2$.

$x=1,\ y=3$ is an obvious solution. Samāsa on (1, 3, 1) by itself gives $x=6,\ y=17$ as another solution. Samāsa between (1, 3, 1) and (6, 17, 1) gives a third solution $x=35,\ y=99$, and so on.

(4) $11x^2+1=y^2$.

Solutions : $x=3,\ y=10$; $x=60,\ y=199$, and so on.

All the above are from Bhaskara.

(5) $92x^2+1=y^2$ (Brahmagupta)

$x=120,\ y=1151$. [Start with $92\cdot1^2+8=100$.]

(6) $83x^2+1=y^2$ (Brahmagupta).

Here $\qquad 83\cdot1^2-2=9^2$

$$x=9,\ y=82.$$

89. The following two examples from Narayana who wrote his *Bijaganita* in 1350 are good illustrations of the Chakravala process.

(7) $103x^2+1=y^2$

Here $\qquad 103.1^2-3=10^2$.

We choose $m+10$ to be divisible by -3. We take $m=11$, getting $103.7^2-6=71^2$.

We next choose n so as to make $7n+71$ to be divisible by -6. We take $n=7$.

$$\therefore \quad 103.20^2+9=203^2.$$

We next choose p so as to make $20p+203$ to be divisible by 9. We take $p=11$, which gives

$$103.47^2+2=477^2$$

which is in the required form. Brahmagupta's method now readily gives $x=22419$, $y=227528$.

(8) $97x^2+1=y^2$.

Here $$97.1^2+3=10^2.$$

$m+10$ is divisible by 3, by taking $m=11$. Hence we get

$$97.7^2+8=69^2.$$

$7n+69$ is divisible by 8 if $n=13$. Hence we get

$$97.20^2+9=197^2.$$

$20p+197$ is divisible by 9 if $p=14$. Hence we get

$$97.53^2+11=522^2.$$

$53q+522$ is divisible by 11, if $q=8$. Hence we get

$$97.86^2-3=847^2$$

$86r+847$ is divisible by 3, if $r=10$. Hence we get

$$97.569^2-1=5604^2,$$

which is in the required form. The Samāsa between

	569	5604	-1
and	569	5604	-1

gives tne solution. $x=6,377,352$, $y=62,809,633$.

90. The determination of integral solutions of the equation $Nx^2+1=y^2$ is the fundamental problem, and is of great interest. The determination of *rational* solutions of this and of some other equations has received some attention on the part of Indian mathematicians, and is not without interest.

We have the identity

$$N.1^2+(m^2-N)=m^2$$

Performing Samāsa on $1\ m\ m^2-N$ by itself, we obtain

$$N(2m)^2+(m^2-N)^2=(m^2+N)^2.$$

Hence, if m is any rational number, $x=\pm\frac{2m}{m^2-N}$, $y=\pm\frac{m^2+N}{m^2-N}$ gives a rational solution of the given equation. This solution is due to Sripati (1039).*

91. *The equation* $Nx^2\pm c=y^2$. If (p, q) be a rational solution, obtained by any process whatsoever, and if (α, β) be a solution of $Nx^2+1=y^2$, then by the Principle of Samāsa,

$$x=p\beta\pm q\alpha, \qquad y=q\beta\pm Np\alpha$$

is a solution of the given equation. We can repeat the process again on this solution, and thus we can get an infinite number of solutions [B.S.S., xviii, 66].

Hence, *if the equation* $Nx^2\pm c=y^2$ *admits of one rational (or integral) solution, it has an infinite number of rational (or integral) solutions.*

For the equation $13x^2+17=y^2$, Narayana gets the values of p, q as follows :

$x=1, y=4$ is a solution of $13x^2+3=y^2$

$x=1, y=8$ is a solution of $13x^2+51=y^2$

Performing Samāsa between (1 4 3) and (1 8 51), we obtain $x=12, y=45$ as solutions of $13x^2+153=y^2$.

Hence $x=4, y=15$ are solutions of $13x^2+17=y^2$.

92. The equation $Mn^2x^2\pm c=y^2$ [B.S.S., xviii, 70]. This is at once transformed into $Mu^2\pm c=y^2$, by putting $nx=u$. Hence if (u, y) be a rational solution of the latter, $\left(\frac{u}{n}, y\right)$ is a rational solution of the former.

93. *The equation* $a^2x^2\pm c=y^2$.

Writing this as $(y-ax)(y+ax)=\pm c$, put

$$y-ax=m$$

$$y+ax=\pm c/m.$$

Hence, $$x=\frac{1}{2a}\left(\pm\frac{c}{m}-m\right), \quad y=\tfrac{1}{2}\left(\pm\frac{c}{m}+m\right)$$

* *Siddhanta-Sekhara*, xiv, 33.

are rational solutions, if m is any rational number [B.S.S., xviii, 69]. Brahmagupta gives the following examples :

(a) $9x^2+52=y^2$.

Hence, $$x=\frac{1}{6}\left(\frac{52}{m}-m\right),\quad y=\frac{1}{2}\left(\frac{52}{m}+m\right).$$

Taking $m=2$, we get the integral solution (4, 14).

(b) $4x^2+33=y^2$.

$$x=\frac{1}{4}\left(\frac{33}{m}-m\right),\quad y=\frac{1}{2}\left(\frac{33}{m}+m\right)$$

$m=1$ gives the integral solution (8, 17). $m=3$ gives the solution (2, 7).

94. *The equation* $Nx^2-k^2=y^2$. Bhaskara has observed that *a rational solution of* $Nx^2-1=y^2$ *is impossible, unless* N *is the sum of the squares of two rational numbers.* The result is true more generally for the equation $Nx^2-k^2=y^2$. For, if $x=\frac{p}{q}$, $y=\frac{r}{s}$ be a solution,

$$N\left(\frac{p}{q}\right)^2=k^2+\left(\frac{r}{s}\right)^2$$

$$\therefore\quad N=\left(\frac{kq}{p}\right)^2+\left(\frac{rq}{ps}\right)^2.$$

When $N=m^2+n^2$, Bhaskara gives $\left(\frac{1}{m}, \frac{n}{m}\right)$ and $\left(\frac{1}{n}, \frac{m}{n}\right)$ as two rational solutions of $Nx^2-1=y^2$. Similarly $\left(\frac{k}{m}, \frac{kn}{m}\right)$, $\left(\frac{k}{n}, \frac{km}{n}\right)$ are solutions of $(m^2+n^2)x^2-k^2=y^2$.

As an example, Bhaskara solves the equation $13x^2-y^2=1$ in three different ways.

(i) Since $13=2^2+3^2$, two rational solutions as given above are $(\frac{1}{2}, \frac{3}{2})$ and $(\frac{1}{3}, \frac{2}{3})$.

(ii) A solution of $13x^2-4=y^2$ is (1, 3). Hence, dividing by 4, a solution of $13x^2-1=y^2$ is $(\frac{1}{2}, \frac{3}{2})$.

(ii) Similarly from the solution (1, 2) of the equation $13x^2-9=y^2$, a solution of $13x^2-1=y^2$ is obtained, viz. $(\frac{1}{3}, \frac{2}{3})$.

By using Bhaskara's cyclic method, integral solutions can be

obtained, as has been observed by Datta and Singh.* Following the method explained in §86,

$$13\left(\frac{\frac{m}{2}+\frac{3}{2}}{-1}\right)^2+\frac{m^2-13}{-1}=\left(\frac{\frac{3m}{2}+\frac{13}{2}}{-1}\right)^2$$

or,
$$13\left(\frac{m+3}{-2}\right)^2+\frac{m^2-13}{-1}=\left(\frac{3m+13}{-2}\right)^2$$

A convenient value for m is $m=3$. This gives

$$13.3^2+4=11^2$$

Repeating the process,

$$13\left(\frac{3n+11}{4}\right)^2+\frac{n^2-13}{4}=\left(\frac{11n+39}{-4}\right)^2$$

Taking $n=3$, we obtain $13.5^2-1=18^2$.

Hence an integral solution of $13x^2-1=y^2$ is (5, 18).

The method of continued fractions gives this as the smallest solution in positive integers. This very example will be found in books on algebra in the chapter on continued fractions. By way of comparison with the Indian method, we write down below, the method by continued fractions.

We have

$$\sqrt{13}=3+\underset{*}{\frac{1}{1+}}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\underset{*}{\frac{1}{6}}.$$

The number of elements in the recurring cycle is 5.

Hence (p_5, q_5) is the smallest solution of the equation $x^2-13y^2=-1$, i.e. of $13y^2-1=x^2$. We have by calculation $p_5=18$, $q_5=5$. The general solution is $p_{(2t-1)c}$, $q_{(2t-1)c}$, t being any integer. See Barnard and Child : *Higher Algebra* p. 535, also Chrystal, *Algebra*, Part II, p. 482.

95. Having made his contribution to the solution of the equation $Nx^2+1=y^2$, thereby supplementing Brahmagupta's beautiful work on this, Bhaskara now revels in the solution of more general equations of the second degree, by using very clever devices to reduce them to the form $Nx^2+1=y^2$. Considering the times when he lived, and the general level of mathematics all over the

* *History of Hindu Mathematics*, pt. II, p. 180.

world at that time, the ingenuity that he has exhibited is remarkable, and would give him a place of honour in the world history of mathematics. We deal with a number of such equations. The methods indicated by Bhaskara lead however to rational solutions only, though in the concrete problems that he has given, it is possible to get integral solutions.

Type 1. To find x such that $6x^2+2x=$ a square y^2.

Bhaskara's solution :

$$6(6x^2+2x)+1=6y^2+1$$

or
$$(6x+1)^2=6y^2+1$$

We solve $6y^2+1=u^2$, where $u=6x+1$. The Varga-Prakriti method described in the preceding pages gives $y=2$, $u=5$, or $y=20$, $u=49$ as two solutions. The first gives $x=\frac{2}{3}$, while the second gives $x=8$. In this way, an infinite number of solutions are obtained.

By describing the method to be applied in general terms, Bhaskara makes it plain that he is dealing with a general type, and not a manipulative example. In modern symbology, the general type here is the equation

$$ax^2+bx+c=y^2.$$

Bhaskara's method transforms this to

$$(ax+\tfrac{1}{2}b)^2=ay^2+\tfrac{1}{4}b^2-ac.$$

Putting $ax+\frac{1}{2}b=z$, and $\frac{1}{4}b^2-ac=k$, we have

$$ay^2+k=z^2.$$

If (α, β) be one solution of this, while (p, q) is a solution of $ay^2+1=z^2$, *Samāsa* gives $y=\alpha q\pm\beta p$, $z=\beta q\pm a\alpha p$ as the general solution. From the value of z, we obtain the value of x.

Type 2. The equation

$$ax^2+bx+c=a'y^2+b'y+c'.$$

As an illustration, Bhaskara proposes the problem :

The first term of an arithmetic progression is 3, and the common difference is 2. Three times the sum of a certain number of terms is equal to the sum of another number of terms.

Let three times the sum of x terms = sum of y terms.

$$\therefore \quad 3(x^2+2x)=y^2+2y.$$

We transform this to

$$(3x+3)^2=3y^2+6y+9$$

$$\therefore \quad 3y^2+6y+9=u^2, \text{ which is of type 1.}$$

$$\therefore \quad 3u^2-18=9y^2+18y+9=(3y+3)^2=t^2, \text{ say.}$$

The solutions for u, t are $u=9$, $t=15$ and $u=33$, $t=57$.

Hence $x=2, y=4$, or $x=10, y=18$.

In the general case, we write

$$(ax+\tfrac{1}{2}b)^2=aa'y^2+ab'y+(ac'+\tfrac{1}{4}b^2-ac),$$

and then proceed as in Type 1, writing $ax+\frac{1}{2}b=u$.

Type 3. The equation $ax^2+by^2+c=z^2$.

Bhaskara adopts several artifices, to solve this equation.

(1) Put $x=my$. We obtain

$$z^2=(am^2+b)\,y^2+c=\alpha y^2+c,$$

where $\alpha=am^2+b$. The equation is reduced to the type treated in §91.

(2) Put $x=my+n$. The equation reduces to

$$z^2=\alpha y^2+2amny+\beta,$$

where $\alpha=am^2+b$, and $\beta=an^2+c$.

Completing the square on the right side, the equation is reduced to the same form as in (1).

(3) Put $by^2+c=w^2$. Hence $z^2-w^2=ax^2$

Put $z-w=mx$, $z+w=\dfrac{a}{m}x$.

$$\therefore \quad z=\frac{1}{2}\left(m+\frac{a}{m}\right)x, \quad w=\frac{1}{2}\left(\frac{a}{m}-m\right)x.$$

From any solution of $by^2+c=w^2$ (§91), we thus obtain a solution for x and y of the given equation.

(4) If $c=0$, the equation is $ax^2+by^2=z^2$.

Putting $x=uy$, $z=vy$, the equation reduces to

$$au^2+b=v^2.$$

All these artifices thus lead to the equation of the type $Nx^2\pm k=y^2$. When $c=0$, the equation is homogeneous in x, y, z and hence admits of integral solutions.

The following concrete examples are given by Bhaskara (*Bijaganita*, Sl. 187).

Tell two numbers, the sum of whose squares respectively multiplied by 7 and 8 yields a square, and their difference does so, when added to one.

In other words, solve the equations

$$(1)\quad 7x^2+8y^2=z^2$$

$$(2)\quad 7x^2-8y^2+1=t^2.$$

Putting $x=uy$, $z=vy$, (1) reduces to $7u^2+8=v^2$. $u=2$, $v=6$ is a solution. Hence $x:y:z=2:1:6$ is a solution.

To solve (2), Bhaskara takes $x=2y$, getting $20y^2+1=t^2$. $y=2$, $t=9$, and $y=36$, $t=161$ are two solutions. Hence $x=4$, $y=2$ and $x=72$, $y=36$ are solutions.

Type 4. The equation $ax^2+bxy+cy^2=z^2$.

If a is a perfect square p^2 say, we write this as

$$\left(px+\frac{by}{2p}\right)^2=z^2-y^2\left(c-\frac{b^2}{4p^2}\right)$$

or

$$z^2-w^2=y^2\left(c-\frac{b^2}{4p^2}\right), \text{ where } w=px+\frac{by}{2p}.$$

Putting

$$z-w=my,\ z+w=\left(c-\frac{b^2}{4p^2}\right)\frac{y}{m},$$

we obtain z and w, from which we can get the values of x, y and z.

The same method holds if c is a perfect square. If neither a nor c be a square, we multiply the given equation by a, and write it as

$$(ax+\tfrac{1}{2}by)^2=az^2-\alpha y^2, \text{ where } w=ax+\tfrac{1}{2}by.$$

This becomes $w^2=az^2-\alpha y^2$, where $w=ax+\frac{1}{2}by$. The equation is now of type 3, with $c=0$.

Type 4 admits of solution in integers, since from any rational solution, we can obtain an integral solution by multiplying by suitable factor, the equation being homogeneous in x, y and z.

The example considered by Bhaskara is $x^2+xy+y^2=z^2$.

We have

$$36x^2+36xy+36y^2=36z^2$$

$$\therefore\quad (6x+3y)^2+27y^2=36z^2.$$

Writing $6x+3y=uy$, $z=vy$, this gives

$$u^2+27=36v^2$$

$u=13$, $v=\frac{7}{3}$ is a solution.

Hence $x:y:z=5:3:7$ is a solution.

96. Bhaskara has shown remarkable ingenuity in devising various other problems, some of them involving equations of higher degree, some of them being simultaneous equations, which are all to be solved by reducing the equations to the Varga Prakriti form by suitable devices. The devices can evidently be applied to general forms of equations, but the types are so varied and miscellaneous, that it is not worth formulation in a general way. The problems themselves, taken in their concrete numerical form, will form interesting puzzle-problems, and one can appreciate Bhaskara's manipulative skill in convertng them to the Varga Prakriti form. We give them and their solutions below. They may be taken in any order.

Problem 1. The square of the sum of two numbers added to the cube of their sum is equal to twice the sum of their cubes. Find the numbers [*Bijaganita*, Sl. 178].

Bhaskara mentions that this is an example from ancient authors. This and similar references elsewhere in Bhaskara's work pose a problem for the historian, which remains unsolved—as to the mathematicians and their works who lived after Brahmagupta, but well before Bhaskara, who were well conversant with the Varga Prakriti process, and who might have made their own contributions in this field.

Solution : Take the numbers as $x+y$ and $x-y$. The given condition leads to

$$4x^2+8x^3=2\{(x+y)^3+(x-y)^3\}$$
$$=4x^3+12xy^2$$
$$\therefore \quad 4x^3+4x^2=12xy^2$$
$$\therefore \quad 4x^2+4x+1=12y^2+1.$$

or $$12y^2+1=u^2, \text{ where } u=2x+1.$$

The problem is thus reduced to the Varga Prakriti form. $y=2$, $u=7$; $y=28$, $u=97$ are two solutions. Hence $x=3$, $x=48$ are the corresponding values of x. The required numbers $x+y$, $x-y$ are

therefore (5, 1), or (76, 20). We can get an infinite number of solutions in integral form.

Problem 2. Find x such that $5x^4-100x^2=$ a perfect square y^2. [*Bijaganita*, Sl. 181].

The general form is $ax^{2n+2}+bx^{2n}=y^2$, which is reduced to $y^2=x^{2n}z^2$, by putting $ax^2+b=z^2$. Hence if (x, z) is a solution of the latter, the corresponding y is given by $x^n z$. In the present problem, the equation $5x^2-100=z^2$ has the solutions (10, 20), (170, 380), etc. The corresponding values of y are 200, 64600.

Problem 3. To find two numbers x and y such that $x-y=$ a square z^2 say, and $x^2+y^2=$ a cube. [*Bijaganita*, Sl. 182].

Bhaskrara puts $x^2+y^2=(z^2)^3$, so that

$$x^2+(x-z^2)^2=z^6$$
$$\therefore \quad 2z^4-4xz^2+4x^2=2z^6$$
$$\therefore \quad 2z^6-z^4=(z^2-2x)^2=t^2,$$

say, which is of the general form given in Problem 2. For the equation $2z^2-1=u^2$, we have the solutions for (z, u), viz. (5, 7), (29, 41), etc. Then $z^2-2x=\pm z^2u$.

Hence, when $z=5$, $u=7$, $x=100$, $y=75$
when $z=29$, $u=41$, $x=17661$, $y=16820$.

We can thus obtain an infinite number of integral solutions, but the method is theoretically imperfect, for we are really solving the equation

$$x^2+y^2=(x-y)^3, \text{ where } x-y=z^2.$$
$$\text{i.e.} \quad (y+z^2)^2+y^2=u^3.$$

Narayana [*Ganita Kaumudi* (1350), 1—58] writes down the rational solutions of a number of equations of this type. The equations are

(i) $x^3+y^3=x^2+y^2$ (ii) $x^3+y^3=(x-y)^2$ (iii) $x^3+y^3=xy$.
(iv) $(x+y)^3=x^2+y^2$ (v) $(x+y)^3=(x-y)^2$ (vi) $(x+y)^3=xy$.

For (i),

$$x=\frac{(m^2+n^2)m}{m^3+n^3}, \quad y=\frac{(m^2+n^2)n}{m^3+n^3}$$

similarly, for (iv),

$$x=\frac{m^2n}{m^3+n^3}, \quad y=\frac{mn^2}{m^3+n^3}.$$

In the language of analytical geometry, the problem is identical with the parametric representation in rational form of these curves, which are all universal plane cubics. (iii) is the folium of Descartes. Putting $y=tx$, and then $t=n/m$, the solutions of all these are obtained in terms of two rational numbers m and n. Narayana's solution is identical (without any smell of analytical geometry) with the usual geometrical method.

Problem 4. Find two numbers x and y such that $x+y$ is a square, say t^2 while x^2+y^3 is also a square, say z^2. [*Bijaganita*, Sl. 188.]

Solution. Writing the second equation as $z^2-x^2=y^3$, we can put $z-x=\lambda$, $z+x=y^3/\lambda$, so that

$$z=\tfrac{1}{2}\left(\frac{y^3}{\lambda}+\lambda\right), \quad x=\tfrac{1}{2}\left(\frac{y^3}{\lambda}-\lambda\right).$$

Putting $\lambda=y$, we write this as

$$x=\tfrac{1}{2}(y^2-y), \; z=\tfrac{1}{2}(y^2+y)$$

$$\therefore \quad x+y=\tfrac{1}{2}(y^2+y)=t^2$$

$$\therefore \quad 4y^2+4y+1=8t^2+1.$$

We thus reduce the problem to the Varga-Prakṛiti form

$$8t^2+1=u^2, \quad \text{where } u=2y+1.$$

Two solutions for t, u are (6, 17) and (35, 99). The corresponding values for y are 8 and 49. The values of $x=t^2-y$ are respectively 28, 1176.

A second method. Put $x=7u^2$, $y=2u^2$, so that the first condition that $x+y$ is a square is automatically satisfied. The other equation gives

$$8u^6+49u^4=z^2,$$

which comes under the type mentioned in Problem 2. Solutions for u for the equation $8u^2+49=v^2$ are $u=2$, 3, 7, etc. The corresponding values of x and y are (28, 8), (63, 18), (343, 98).

The solutions obtained by the two methods are not necessarily the same, for the second method gives the ratio $x:y=7:2$ which is possessed only by the solution (28, 8) obtained by the first method.

Problem 5. To find a number x such that $3x+1$ and $5x+1$ are both squares. [*Bijaganita*, Sl. 197]

Solution. Let $3x+1=u^2=(3y+1)^2$

$$\therefore \quad x=3y^2+2y$$

$$\therefore \quad 5x+1=15y^2+10y+1=v^2$$

$$\therefore \quad 15v^2+10=225y^2+150y+25$$

$$=(15y+5)^2$$

We have thus $\quad 15v^2+10=t^2.$

Two solutions for (v, t) are (9, 35), (71, 275). Hence the corresponding values of y are 2 and 18.

$$\therefore \quad x=3y^2+2y=16, \ 1008.$$

Otherwise, we write $3x+1=u^2$

$$\therefore \quad 5x+1=\tfrac{5}{3}u^2-\tfrac{2}{3}=v^2$$

$$\therefore \quad \tfrac{5}{3}(u^2-1)+1=v^2$$

$$\text{or} \quad (5u)^2=15v^2+10=t^2,$$

which is the equation obtained previously.

Bhaskara has treated this in a more general problem. The problem occurs in particular forms in earlier works, and is hence of sufficient interest to be dealt with at greater length.

The general problem is to find integral or rational solutions of the simultaneous equations $ax+c=u^2$, $bx+d=v^2$. A particular case occurs in the Bakhshāli manuscript (vide Chapter III) and in Brahmagupta in the form $x+a=u^2$, $x+b=v^2$. [Brahmasphuta Siddhanta, xvii, 74].

We haue $u^2-v^2=a-b$. Hence, putting $u-v=m$, $u+v=\dfrac{a-b}{m}$, we have Brahmagupta's solution

$$x=\left\{\frac{1}{2}\left(\frac{a-b}{m}+m\right)\right\}^2-a, \text{ or } x=\left\{\frac{1}{2}\left(\frac{a-b}{m}-m\right)\right\}^2-b.$$

The problem in the Bakhshāli work corresponds to the values $a=5$, $b=-7$, and the solution given is that obtained by taking $m=2$ in the general solution.

For the general problem $ax+c=u^2$, $bx+d=v^2$, Bhaskara [*Bijaganita*, Sl. 195–6] reduces the problem to one of Varga Prakriti. His enunciation may be interpreted thus :

Put $\quad u=mw+\alpha$

$$\therefore \quad x=\frac{(mw+\alpha)^2-c}{a}.$$

The other equation then gives

$$\frac{b}{a}\{(mw+\alpha)^2-c\}+d=v^2$$

or
$$bu^2+(ad-bc)=av^2,$$

which is the extended Varga-Prakriti form (§91). Writing this as

$$abu^2+k=t^2,$$

where $k=a(ad-bc)$ and $t=av$, if (r, s) be one solution for (u, t) and (p, q) one solution of $abu^2+1=t^2$, then $u=qr+ps$, $t=qs+abpr$ is also a solution. We can now write down the value of x. A rational solution for p, q is given by

$$p=\frac{2\lambda}{\lambda^2-ab}, \quad q=\frac{\lambda^2+ab}{\lambda^2-ab}.$$

To obtain this, we put $q=\sec\theta$, whence $p\sqrt{ab}=\tan\theta$. Writing $\tan\frac{\theta}{2}=\frac{t}{\sqrt{ab}}$ we obtain the given values of p and q.

When $c=d=1$, the equations become $ax+1=u^2$, $bx+1=v^2$, for which Brahmagupta [B.S.S., xviii, 78] gives the solution

$$x=\frac{8(a+b)}{(a-b)^2}, \quad u=\frac{3a+b}{a-b}, \quad v=\frac{a+3b}{a-b}.$$

Problem 6. To solve in integers

(i) $x^2+y^2+1=u^2$, $\quad x^2-y^2+1=v^2$.

Similarly, to solve

(ii) $x^2+y^2-1=u^2$, $\quad x^2-y^2-1=v^2$

[Bhaskara, *Bijaganita*, Sl. 194]

Putting $y^2=4z^2$, $x^2=5z^2-1$, the equations (i) are satisfied. Two solutions of $5z^2-1=x^2$ are $z=1$, $x=2$, and $z=17$, $x=38$. The corresponding values of y are 2 and 34.

Two solutions of (i) are therefore $x=2, y=2$ and $x=38, y=34$.

To solve (ii), we put $y^2=4z^2$, $x^2=5z^2+1$, and proceed similarly. Two solutions for (x, z) are (9, 4) and (161, 72). Hence the corresponding solutions for x, y are (9, 8) and (161, 144).

Bhaskara explains the general method of solution.

For equation (i), put $y^2 = 4m^2z^2$.

We have $\quad 2y^2 = u^2 - v^2 = 8m^2z^2$.

Put $\quad u - v = 2mz, \quad u + v = 4mz \quad \therefore \quad u = 3mz$.

This gives $\quad x^2 + 1 = 5m^2z^2$.

m may be chosen as any convenient integer. Similarly, we can put $u - v = \lambda z$ where λ is any even factor of $8m^2$ such that u turns out to be a multiple of z.

Otherwise, assume $x^2 + 1 = a^2 + b^2, y^2 = 2ab$. To make y^2 a perfect square, put $a = \lambda^2 z$, $b = \frac{1}{2}\mu^2 z$ so that $y^2 = \lambda^2\mu^2z^2$. For example, take $\lambda^2 = 9$, $\mu^2 = 4$. Then $y^2 = 36z^2$, $x^2 + 1 = 85z^2$.

We have only to write $x^2 - 1$ instead of $x^2 + 1$, to solve equation (ii). To solve for x and y finally, we must solve the Varga-Prakriti equation

$$x^2 \pm 1 = (\lambda^4 + \tfrac{1}{4}\mu^4)z^2.$$

Bhaskara has also given this problem in the *Lilavati* (sll. 59, 60, 61) as follows :

To find numbers x, y such that $x^2 + y^2 - 1$ and $x^2 - y^2 - 1$ are both squares.

Bhaskara has simply given the answers, without explaining the method, for the obvious reason that in a book devoted to arithmetic, the method of solution, depending on Varga-Prakriti cannot be given. The answers given are

(i) $\quad x = \dfrac{1}{2}\left(\dfrac{8n^2 - 1}{2n}\right)^2 + 1, \quad y = \dfrac{8n^2 - 1}{2n}$

(ii) $\quad x = n + \dfrac{1}{2n}, \quad y = 1$

(iii) $\quad x = 8n^4 + 1, \quad y = 8n^3$.

These answers are to be obtained as follows :

Writing $\quad x^2 + y^2 - 1 = u^2, \quad x^2 - y^2 - 1 = v^2$,

we have

$$u^2 - v^2 = 2y^2, \text{ or } \left(\frac{u}{y}\right)^2 = \left(\frac{v}{y}\right)^2 + 2.$$

A solution is given by

$$\frac{u}{y} = t + \frac{1}{2t}, \quad \frac{v}{y} = t - \frac{1}{2t}.$$

Putting $y=mz$, we have

$$u=\left(t+\frac{1}{2t}\right)mz, \quad v=\left(t-\frac{1}{2t}\right)mz, \quad y=mz.$$

Put $mz=1$. We obtain the solution (ii).

Put $m=2t$, $z=4t^3$. Then $u=8t^3\left(t+\frac{1}{2t}\right)$

$\therefore$ $x^2=u^2-y^2+1=(8t^4+1)^2$. This is the solution (ii).

To obtain (i), change t to $2n$

$$\therefore \quad u=\frac{8n^2+1}{4n}.mz, \quad v=\frac{8n^2-1}{4n}mz.$$

Put $m=8n^2-1$, $z=\frac{1}{2n}$

$$\therefore \quad y=\frac{8n^2-1}{2n}, \quad u=\frac{64n^4-1}{8n^2},$$

whence x is to be obtained.

The details employed by Bhaskara are not clear, though evidently he has used convenient solutions for the Varga-Prakriti equation. The method suggested by Datta and Singh* is rather complicated, to be ascribed to Bhaskara.

Problem 7. To solve the simultaneous equations $ax^2+by^2=u^2$, $a'x^2+b'y^2+c'=v^2$.

This has already been treated under Type 3, §95 when we considered a numerical example proposed by Bhaskara. Putting $x=my$, $u=ny$, we get

$$am^2+b=n^2.$$

If m, n are solutions of this equation, the second equation gives

$$(a'm^2+b')y^2+c'=v^2.$$

Any rational solution of this (§91) leads to the solution of the given equations. Bhaskara suggests that the same method can be employed to solve the equations

$$ax^2+by^2+c=u^2, \quad a'x^2+b'y^2+c'=v^2.$$

Problem 8. To solve the simultaneous equations

$$a(x^2-y^2)+b=u^2, \quad a'(x^2-y^2)+b'=v^2.$$

* *History of Hindu Mathematics* (1962), Part II, pp. 269-70.

We first reduce the equations to the general form under Problem 5, by putting $x^2-y^2=t$. We shall consider the numerical example given by Bhaskara [*Bijaganita*, Sll. 199—200].

$$2(x^2-y^2)+3=u^2, \quad 3(x^2-y^2)+3=v^2$$

So, we first consider the equations

$$2t+3=u^2, \quad 3t+3=v^2$$

Eliminating t,

$$3u^2=2v^2+3$$

$$\therefore \quad 6v^2+9=9u^2=w^2, \text{ where } w=3u$$

Solutions for v, w are (6, 15), (60, 147). The corresponding values of $t=\frac{1}{3}(v^2-3)$ are 11, 1199.

If we take $x^2-y^2=11$, we choose $x-y=1$, $x+y=11$. This gives $x=6, y=5$.

Similarly, taking $x^2-y^2=1199$, and choosing $x-y=1$, $x+y=1199$, we obtain the solution $x=600$, $y=599$. Choosing $x-y=11$, we obtain $x=60, y=49$.

Problem 9. Solve the equations

$$x^2+xy+y^2=z^2$$

$$(x+y)z+1=v^2.$$

We have already considered the first of these equations as illustrating Type 4, §95. Writing it as

$$36(x^2+xy+y^2)=36z^2,$$

it becomes

$$u^2+27=36v^2,$$

where

$$6x+3y=uy, \quad z=vy.$$

$u=13$, $v=\frac{7}{3}$ is a solution. This gives $x=\frac{5}{3}y$, $z=\frac{7}{3}y$.

The second equation now gives

$$\tfrac{56}{9}y^2+1=u^2,$$

$$\text{or} \quad 56y^2+9=w^2, \text{ where } w=3u.$$

Two solutions for (y, w) are (6, 45), (180, 347). Hence the corresponding solutions for (x, y, z) are (10, 6, 14), (300, 180, 420).

Problem 10. Find two numbers x and y (other than 7 and 6), such that $x-y+3$, $x+y+3$, x^2+y^2-4, x^2-y^2+12 are all square u^2, v^2, r^2, s^2 say, $\frac{1}{2}xy+y$ is a cube p^3, while $u+v+r+s+p+2$ is also a square q^2. [*Bijaganita*, Sl. 193].

This is a complicated problem, and the solution is not straightforward. Let us consider the general form $x-y+k=u^2$, $x+y+k=v^2$, $x^2-y^2+k'=s^2$, $x^2+y^2+k''=t^2$. Bhaskara puts $u=w-\alpha$, where α is a convenient number, and then states that $v=(w-\alpha)+\sqrt{k'/k}$. This is obtained as follows :

$$x^2-y^2+k'=(u^2-k)(v^2-k)+k'$$
$$=u^2v^2-k(u^2+v^2)+k^2+k'.$$

A value of k' which makes the right side a perfect square is $k'=k(v-u)^2$.

Hence, $\qquad v=u+\sqrt{k'/k}.$

Solving

$$x-y=(w-\alpha)^2-k$$
$$x+y=(w-\alpha+\sqrt{k'/k})^2-k$$

for x and y, the last equation gives on simplification

$$u^4+2\gamma u^3+(3\gamma^2-2k)u^2+2\gamma(\gamma^2-k)u+\tfrac{1}{2}k^2+\tfrac{1}{2}(\gamma^2-k)^2+k''=t^2,$$

where $\gamma=\sqrt{k'/k}$. A solution for u has to be obtained by trial.

In the numerical problem given, Bhaskara takes $u=w-1$. We obtain

$$x-y=(w-1)^2-3=w^2-2w-2$$
$$x+y=(w-1+2)^2-3=w^2+2w-2, \text{ since } \gamma=2.$$

$\therefore \quad x=w^2-2, y=2w.$ These values give

$$x^2-y^2+12=(w^2-4)^2,\ x^2+y^2-4=w^4,\ \tfrac{1}{2}xy+y=w^3.$$

Thus all the equations except the last one are satisfied for all values of w. Tbe last equation gives

$$2w^2+3w-2=q^2.$$

Completing the square, we write this as

$$(4w+3)^2=8q^2+25$$

which is an equation of the Varga-Prakriti form. By the usual method, two of the solutions are $q=5$, $4w+3=15$ and $q=175$, $4w+3=495$. Hence two solutions for (x, y) are (7, 6), (15127, 246).

We could also take $\alpha=0$. We then write $x-y+3=w^2$, $x+y+3=(w+2)^2$. Hence,

$$x=w^2+2w-1,\ y=2w+2.$$

The only equation to be satisfied is

$$2w^2+7w+3=q^2, \quad \text{or} \quad (4w+7)^2=8q^2+25.$$

w here is $w+1$ of the previous work. Hence we obtain solutions for x and y as before.

Bhaskara calls this example as कस्याप्युदाहरणम् (example given by some one). In other words, this is an example taken from some previous writer.

Another example taken from "an ancient author" is the following :

Problem 11. If

$$\sqrt[3]{\tfrac{1}{2}(xy+y)}+\sqrt{x^2+y^2}+\sqrt{x^2-y^2+8}+\sqrt{x+y+2}+\sqrt{x-y+2}=q^2,$$

find solutions for x and y other than 8 and 6 [*Bijaganita*, Sl. 190].

This is solved by choosing x and y so that all the surds reduce themselves. Hence we put

$$x-y+2=u^2,\ x+y+2=v^2,\ x^2-y^2+8=s^2,\ x^2+y^2=t^2,$$

$$\tfrac{1}{2}(xy+y)=p^3,\ u+v+s+t+p=q^2.$$

Now the problem is similar to Problem 10. Proceeding as before, we put $u=w-1$, and obtain $x=w^2-1$, $y=2w$. All the equations except the last are identically satisfied. The last gives

$$2w^2+3w-2=q^2, \quad \text{or} \quad (4w+3)^2=8q^2+25,$$

the same equation as was obtained in Problem 10. Solutions of this are

$$\left.\begin{array}{r} q=5 \\ 4w+3=15 \end{array}\right\}, \qquad \left.\begin{array}{r} q=30 \\ 4w+3=85 \end{array}\right\}, \qquad \left.\begin{array}{r} q=175 \\ 4w+3=495 \end{array}\right\}, \text{ etc.}$$

Hence solutions for (x, y) are $(8, 6)$, $(\frac{1677}{4}, 41)$, $(15128, 246)$, etc.

We may take other values for α (see Problem 10), but the solution is not materially different.

Chapter XI

MISCELLANEOUS EQUATIONS

97. All the equations in the previous chapter ultimately depend on the solution of a Varga Prakriti equation. There are a few other equations in the works of Bhaskara and of earlier writers, which are elementary and do not involve the Varga Prakriti process. These will be treated briefly in this chapter.

98. *The equation* $ax^2+c=by$. [The *Varga-Kuttaka.*]

Bhaskara [*Bijaganita*, sll. 202—5] proposes the following problem.

For what value of x is (i) x^2-4 divisible by 7 (ii) x^2-30 divisible by 7.

For (i), $x^2=7y+4$. The substitution $y=7z^2+4z$ makes the right side a perfect square. We get

$$x^2=(7z+2)^2.$$

Hence $x=7z+2$ is the general solution, where z is any integer.

In the case of (ii), 30 is not a square number. We write

$$x^2=7y+30=(7z+k)^2,$$

where k is to be chosen so that y is an integer.

$$\therefore \quad k^2-30\equiv 0 \pmod 7.$$

$k=11$ is a solution. $\therefore$ $x=7z+11$, $z=0, \pm 1, \pm 2, \ldots\ldots$ gives the general solution for x in integers. Bhaskara explains the solution as follows: 30 being a non-square, divide 30 by 7, getting the remainder 2. Add to this a multiple of 7 so as to make it a square. Thus $2+14=4^2$. Hence $x^2=7y+30=(7z+4)^2$.

$$\therefore \quad x=7z+4, \text{ where } z \text{ is any integer.}$$

In the general form, we therefore consider two cases :

(i) $x^2=by+c^2$. We put $y=bz^2+2c$, getting

$$x^2=(bz+c)^2 \qquad \therefore \quad x=bz+c.$$

We thus get integral solutions for x and y, in terms of an arbitrary parameter z.

(ii) $x^2=by+c$, where c is not a square.

We write $x^2=(bz+k)^2 \quad \therefore \quad k^2-c\equiv 0 \pmod{b}$.

Hence *the given equation has solutions, if and only if c or $c-\lambda b$ is a quadratic residue of b.*

More generally, we consider the equation

$$ax^2=by+c$$

$$\therefore \quad a^2x^2=aby+ac=by'+ac, \text{ where } y'=ay.$$

Let $by'+ac=(kbz+k')^2$, where k' is so chosen that $k'^2\equiv ac \pmod{b}$. An integral solution for y' exists if ac (after reduction mod b) is a quadratic residue of b.

Then $$ax=kbz+k'$$

which can be solved by the Kuttaka method. (Chapter IX).

By way of illustration, Bhaskara gives the following problem. (*Bijaganita*, Sl. 207).

Find x such that $5x^2+3$ is divisible by 16.

Let $5x^2+3=16y \quad \therefore \quad 25x^2=80y-15$

$$=16y'-15 \text{ where } y'=5y$$

$\therefore \quad 25x^2=16y'-15=(8z+1)^2$, since this gives an integral solution for y'.

$$\therefore \quad 5x=8z+1.$$

The solution for this is given by $x=8t+5$, $t=0, \pm 1, \pm 2$, etc.

The problem dealt with in this section has been called the *Varga-Kuttaka*, since it resembles the Kuttaka or linear indeterminate equation, except that one term is a square. Simple cases have been solved by Brahmagupta and others by more or less a trial method. Bhaskara has given the method indicated above when c is a square. But his method is not quite satisfactory, when c is a non-square. Writing $c\equiv n \bmod b$, for the equation $x^2=by+c$, he requires s to be determined such that $n+bs=$a square r, and then puts $x=bu+r$. This is the original problem itself, except that s may be obtained by trial, if n is small.

The method has been extended by Bhaskara to higher powers of y. The equation $bx+c=y^3$ has been called the *Ghana-Kuttaka* (cube-Kuttaka).

If $c=\beta^3$, Bhaskara puts $y=bv+\beta$, getting at once

$$x=b^2v^3+3\beta v(bv+\beta).$$

If c is not a cube, let $c=bm+n$. We find s (by trial) such that $n+bs=r^3$.

The solution is given by

$$y=bv+r, \quad x=b^2v^3+3vr(bv+r)-(m-s).$$

Illustration : To solve $5x+6=y^3$. [*Bijaganita*, Sl. 206].

$$b\equiv 1 \pmod 5), \text{ and } 1+43\cdot 5=6^3, \text{ i.e. } s=43, r=6.$$

Putting $y=5v+6$, we get

$$x=25v^3+18v(5v+6)+42,$$

where v is any integer.

In general, if $\quad ay^3=bx+c,$

we have $\quad a^3y^3=a^2bx+a^2c=bx'+a^2c,$

where $\quad x'=a^2x.$

Let $bx'+a^2c=(kbu+k')^3$, where k is any integer, and $k'^3-a^2c\equiv 0$ (mod b). This is solvable if a^2c is a *cubic residue* of b. We then get $ay=kbu+k'$, which can be solved by the Kuttaka method.

99. *Miscellaneous simultaneous equations.*

Example 1. To find numbers x and y such that $x+y$, $x-y$, $xy+1$ are all perfect squares. [*Laghu Bhāskarīya* of Bhaskara I]

Brahmagupta gives the solution [B.S.S. xviii, Sl. 72] as follows :

$$x=k(m^2+n^2), \quad y=k(m^2-n^2), \text{ where}$$

$$k=\frac{(m^2+n^2)+(m^2-n^2)}{[\frac{1}{2}\{(m^2+n^2)-(m^2-n^2)\}]^2}.$$

To obtain this, we take $x=2z^2(m^2+n^2)$, $y=2z^2(m^2-n^2)$, so that $x+y$ and $x-y$ are squares. Then

$$xy+1=4z^4(m^4-n^4)+1$$

should be a square. The right side is

$$(2z^2m^2-1)^2+4z^2(m^2-z^2n^4).$$

The condition is satisfied by putting $m^2=z^2n^4$.

$$\therefore \quad 2z^2=\frac{2m^2}{n^4}=\text{the value of } k \text{ given.}$$

Example 2. To find numbers x and y such that $x+y$ and $x-y$ are squares, while xy is cube. [*Bijaganita*, p. 56]

We take $x=5z^2$, $y=4z^2$, and put $20z^4=$ a cube p^3z^3, say. Hence $z=\frac{p^3}{20}$. We may take p to be a multiple of 10.

Example 3. To find 4 numbers x, y, z, w such that when added to a given number α, each of them gives a square; further xy, yz, zw when added to another given number β also give squares. [*Bijaganita*, p. 68].

In other words, $x+\alpha=p^2$, $y+\alpha=q^2$, $z+\alpha=r^2$, $w+\alpha=s^2$; $xy+\beta=\xi^2$, $yz+\beta=\eta^2$, $zw+\beta=\zeta^2$.

Bhaskara states that the method of solution is taken from an earlier writer, and that the method is well-known.

We have
$$xy+\beta=(p^2-\alpha)(q^2-\alpha)+\beta$$
$$=(pq-\alpha)^2+\{\beta-\alpha(q-p)^2\}.$$

We choose p and q so that $\alpha\,(q-p)^2=\beta$.

$$\therefore\quad q=p\pm\gamma,\ \text{where } \gamma=\sqrt{\beta/\alpha}.$$

Similarly we take $r=q\pm\gamma$, $s=r\pm\gamma$, so that p, q, r, s form an A.P. of common difference γ.

Hence if p is any number, the solutions are

$$x=p^2-\alpha,\quad y=(p\pm\gamma)^2-\alpha,\quad z=(p\pm2\gamma)^2-\alpha,\quad w=(p\pm3\gamma)^2-\alpha.$$

These values evidently satisfy the last three equations also, since

$$xy+\beta=(pq-\alpha)^2,\quad yz+\beta=(qr-\alpha)^2,\quad zw+\beta=(rs-\alpha)^2.$$

Datta and Singh (p. 289) remark that the last three equations may be generalised by taking $xy+\beta_1=\xi^2$, $yz+\beta_2=\eta^2$, $zw+\beta_3=\zeta^2$.

Proceeding as before, we now take

$$q=p\pm\sqrt{\beta_1/\alpha},\quad r=q\pm\sqrt{\beta_2/\alpha},\quad s=r\pm\sqrt{\beta_3/\alpha}.$$

The following numerical example is given by Bhaskara [*Bijaganita*, p. 67].

$x+2$, $y+2$, $z+2$, $w+2$ are squares, xy, yz, zw added to 18 are squares; the square root of the sum of all the roots added to 11 gives 13.

Here $\alpha=2$, $\beta=18$, so that $\gamma=\pm 3$. Hence we have $x=p^2-2$, $y=(p+3)^2-2$, $z=(p+6)^2-2$, $w=(p+9)^2-2$.

Hence,

$$xy+18=\{p(p+3)-2\}^2$$
$$yz+18=\{(p+3)(p+6)-2\}^2$$
$$zw+18=\{(p+6)(p+9)-2\}^2.$$

The last condition of the problem then gives

$$p+(p+3)+(p+6)+(p+9)+(p^2+3p-2)+(p^2+9p+16)$$
$$+(p^2+15p+52)+11=169.$$

i.e. $$3p^2+31p-74=0.$$

Multiplying by 12, this can be written

$(6p+31)^2=43^2$. Hence $p=2$, $q=5$, $r=8$, $s=11$.

The values of x, y, z, w are then 2, 23, 62, 119.

100. *The equation* $axy=bx+cy+d$.

Equations of this type are as old as the Bakhshâli manuscript. The following example and solutions have been preserved in the mutilated text that is available.

$$xy=3x+4y\mp 1.$$

Solution :

$$x=\frac{3.4-1}{1}+4=15$$
$$y=1+3=4$$
$$x=1+4$$
$$y=\frac{3.4+1}{1}+3=16.$$

Evidently, the first is the solution when the last term is -1, and the second when it is $+1$.

The problem is treated by Brahmagupta (xviii, sl. 60) whose rule is taken from an earlier writer, not known now, and which purports to the solution

$$y=\frac{1}{a}\left(\frac{ad+bc}{m}+b\right),\ x=\frac{1}{a}(m+c)$$

where m is any arbitrary number, and where it is assumed that

$b>c$ and $m>\frac{ad+bc}{m}$. These restrictions are however unnecessary, and the rationale of the solution is as follows :

$$axy=bx+cy+d.$$

Multiplying by a, we write it as

$$(ax-c)(ay-b)=ad+bc.$$

Putting $ax-c=m$, we obtain $ay-b=\frac{ad+bc}{m}$, which is the solution given above. We may also put $ay-b=m$. We then get the solutions

$$x=\frac{1}{a}\left(\frac{ad+bc}{m}+c\right),\ y=\frac{1}{a}(m+b).$$

Bhaskara gives the solutions (*Bijaganita*, p. 124) in the following form

$$\left.\begin{aligned} x&=\frac{c}{a}\pm m \\ y&=\frac{b}{a}\pm n \end{aligned}\right\},\quad \text{or}\quad \left\{\begin{aligned} x&=\frac{c}{a}\pm n \\ y&=\frac{b}{a}\pm m \end{aligned}\right.$$

where m and n are factors of $\frac{ad+bc}{a^2}$. The rationale is as follows :

We have $$xy-\frac{b}{a}x-\frac{c}{a}y=\frac{d}{a}$$

$$\therefore\quad \left(x-\frac{c}{a}\right)\left(y-\frac{b}{a}\right)=\frac{d}{a}+\frac{bc}{a^2}.$$

Hence if $\frac{d}{a}+\frac{bc}{a^2}=mn$, we can take $x-\frac{c}{a}=m$, $y-\frac{b}{a}=n$ or $x-\frac{c}{a}=n$, $y-\frac{b}{a}=m$.

The problem is essentially the same as the representation of the hyperbola $axy=bx+cy+d$ rationally in terms of a parameter t.

A different geometrical interpretation and solution is available when $a=1$.

Consider the rectangle $ABCD$ of sides x and y. Cut off $AE=b$, $EG=c$ as in the figure. Let $AB=x$, $BC=y$.

$\therefore$ rect. $AF=bx$, rect. $EH=c(y-b)$.

Since $xy=bx+cy+d$, it follows that rectangle $GC=d+bc$. We therefore choose two factors m and n such that $mn=d+bc$. Then we can take $GH=m$, $GF=n$, or $GH=n$, $GF=m$. Hence $x=m+b$, $y=n+c$, or $x=n+b$, $y=m+c$.

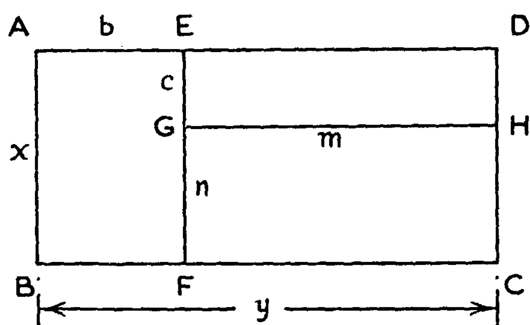


FIG. 20.

If m, n are less than b, c respectively, then $x=c-n$, $y=b-m$ are also solutions, for

$$bx+cy+d=(b-m)(c-n), \quad \text{since } d=mn-bc.$$

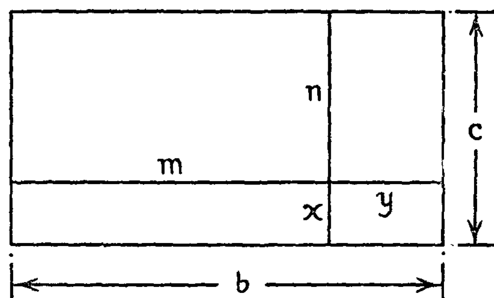


FIG. 21.

In this case, the rectangle xy will stand outside the rectangle mn.

The following numerical examples are given :

(1) $xy=4x+3y+2$. (*Bijaganita*, Sl. 209).

We have $4\times3+2=14=1\times14$, or 2×7.

Hence $x=17, y=5$, or $x=4, y=18$

or $x=10, y=6$, or $x=5, y=11$,

by the rule explained.

(2) $xy=5x+7y-29$. (Sl. 215).

Here d is negative, and $bc+d=6=2\times3$.

$\therefore \begin{matrix}5 & 7\\ 2 & 3\end{matrix}$ and $\begin{matrix}5 & 7\\ 3 & 2\end{matrix}$ i.e. $x=10, y=7$, or $x=9, y=8$ are solutions.

$5-2=y$, $7-3=x$ and $5-3=y$, $7-2=x$ are also solutions.

(3) $xy+3x+5y=62$, or $xy=-3x-5y+62$.

Here b, c are negative. $bc+d=77=7\times11$.

$\therefore$ $7-3=y$, $11-5=x$, or $11-3=y$, $7-5=x$ are solutions.

(4) $\sqrt{(x+y)+xy+(x^2+y^2)}+x+y=23$. (Sl. 211)

A trial method is suggested. Put $y=2$.

Then $x^2+3x+6=(21-x)^2$. Hence $x=\frac{87}{9}$.

Otherwise,

$$x^2+y^2+xy+x+y=(23-x-y)^2.$$

Simplifying,

$$xy=47x+47y-529.$$

Using the rule given,

$$47^2-529=1680=42\times40.$$

But the solutions $47+42=x$, $47+40=y$ are untenable, since $23-x-y$ is to be positive. Hence $x=47-42$, $y=47-40$ are the solutions.

Similarly, if we take 53 in place of 23, the equation becomes

$$xy=107x+107y-2809$$

$$bc+d=107^2-2809=8640=90\times96.$$

$$x=107-96,\ y=107-90 \text{ are solutions.}$$

Chapter XII

INDIAN MATHEMATICS, AFTER BHASKARA

101. We have dealt with, in the previous chapters, the contributions of the leading mathematicians of ancient India. We have incidentally referred here and there to the lesser mathematicians, who were more or less commentators of the works of the great mathematicians. One often finds in the works of these commentators some improvements upon the methods of their leaders, and interesting examples or illustrations here and there. After Bhaskara, for several centuries one comes across only these commentators, except in a remote corner of India, viz., Kerala. The political upheavals that began in this sub-continent soon after Bhaskara, and the consequent absence of security and tranquillity may have been an important factor for the barrenness of scientific activity that prevailed after Bhaskara. India which stood in the front line of mathematical knowledge and research suddenly fell into a state of torpor, and but for one or two outstanding exceptions never recovered from this torpor till the advent of mathematicians like S. Ramanujan trained according to Western methods.

102. One of these exceptions is the great phillip to the study of mathematics and astronomy given by a monarch, Sawai Jayasinha Raja. Students of Indian history will remember the name of Jayasinha, the ablest general who served under Emperor Aurangzeb, and who tried to forge friendship between him and the great Maratha leader Shivaji. While returning after his successful encounter with Shivaji, he took with him a scholar by name Jagannath Pandit, who at the age of twenty, was a profound scholar of Sanskrit. Jayasinha arranged to teach Arabic and Persian to Jagannath Pandit. But soon after returning from the south, Jayasinha died in 1667. His descendent Sawai Jayasinh, or Jayasinha II became the Prince of Amber in Rajasthan in 1699 at the age of thirteen. He had to struggle hard to stabilise himself in his kingdom, and he finally succeeded by 1708. The prince was greatly

interested in mathematics, particularly in astronomy. Jagannath Pandit, now known as Samrat (संराट्) Jagannath was under the benevolence of the prince. He published a translation of the Arabic work *Mijasthi* or *Majisthi*, written by the renowned Nasir Eddin Mahmoud of Persia (died about 1276 A.D.). This work was a translation of Euclid's *Elements*. But Samrat Jagannath's work was not a mere translation, but contained many new proofs not found in Euclid. Some few of these proofs may have been given by Nasir Eddin. This is the first complete translation of Euclid to an Indian language, but some matter from Euclid had found its way in an earlier work, the *Siddhanta Tattva Viveka*, written by Kamalakara, who was an astronomer in the court of the Emperor Jahangir.

Jayasinha Raja published in the name of Emperor Mahmud Shah a book on astronomical tables. From his introduction given to this book, it appears that Jayasinha Raja at quite an early age made a thorough study of the astronomical systems of the Hindus and the Muslims, and of the West, and entertained in his court a number of local and foreign scholars. In order to prepare accurate astronomical tables, he constructed a big astronomical observatory at Delhi, and to confirm the readings obtained here, he erected similar observatories at Jaipur, Mathura, Banaras and Ujjain. The instruments known by the names Jai Prakash, Ram Yantra and Samrat Yanthra were his own inventions. The instruments were of remarkable accuracy, though of huge size. Some years later, Jayasinha Raja learnt that observatories were in existence in Europe. He sent Christian missionaries to Europe, and through them got the tables and some of the instruments made in Europe. There were of course some differences between the European tables and his. Jayasinha's observatories and tables have no scientific value now, but his attempt was commendable. His observatory at Delhi, known by the name *Jantar Mantar* (जंतरमंतर) is today one of the sight-seeing spots in Delhi, and in Jaipur.

The present city of Jaipur was founded by Jayasinha Raja in 1728 A.D. and became his capital. It became a centre for the study of science and art. The Raja was a vassal under the Moghul Emperor Mahmud Shah, and ruled over the provinces of Agra and Malwa. Amidst the political squabbles of the time, in the midst of

wars, the encouragement that he gave to the development of knowledge, and his own achievements in the field deserve the greatest encomium. He died in 1743 A.D. To this day, the name Maharaja Sawai Jaya Sinha is proudly remebered in Rajasthan.

103. The development of mathematics practically came to a full stop, and scholars contended themselves with chewing the cud, studying the works of the great mathematicians, and producing here and there a small bright gem. This remark holds for the entire country, except the state of Kerala in the south-west corner of India. This state includes the former Travancore, Malabar and Cochin. The political upheavals in the rest of India did not very much affect this corner of the country, and the people of this state enjoyed a certain amount of peace and isolation. Some mathematical works of this place dating from the fifteenth to the seventeenth century have recently come to light and contain mathematics of a standard which is startling, and which sets a big puzzle to the historian. In 1835, Charles M. Whish published in Vol. 3 (pp. 509—23) of the *Transactions of the Royal Asiatic Society of Great Britain and Ireland* an article with the title "On the Hindu quadrature of the circle". Whish was an officer of the East India Company's Civil Service at Madras. He refers in his articles to four works, the *Tantra-Sangraham*, *Yukti-Bhāsa*, *Karana Paddhati*, and the *Sadratna-Māla*. Remarks about the dates and authorship of these works will be made presently. Whish writes that the *Tantra-Sangraham* whose date is definitely about 1500 A.D. "laid the foundation for a complete system of fluxions", while the *Sadratna-māla* "abounds with fluxional forms and series to be found in no work of foreign or other Indian countries". They are essentially astronomical treatises but they give in accurate form what we now call Gregory's series, and Euler's series for π, and a number of remarkable rational approximations and rapidly convergent series for π.

Enough internal evidence is available to know the date and authorship of the *Tantra-Sangraha*.* Each chapter concludes with a

* The subject-matter of this topic has been based on an article by C. T. Rajagopal and K. Mukunda Murar in the *Journal of the Royal Asiatic Society* (Bombay Branch), Vol. 20 (1944), 66-82.

statement that the author is Nilkantha (नीलकण्ठ) of Kerala belonging to Gargagôtra. A careful interpretation of the word-numbers used in the slôkas reveals that the work was written in 5 days, from the 26th day of the month Mina in Kali 4601 to the 1st day of Mēsha in Kali 4602. This corresponds to 1502 A.D. The same author wrote *Arya-Bhatiya* of Arya-Bhata (499 A.D.) with a commentary. This has been published in the *Trivandrum Sanskrit Series*. We gather from this that the author was a native of Kunda in Kerala, son of Jātā Veda (जातवेद) and a disciple of Dāmodara, son of Paramēswara who propounded the *Drigganita* system of astronomy in 1353 Saka, i.e. 1430 A.D.

The *Karana-Paddhati* has been printed in the *Trivandrum Sanskrit Series* (No. 20). The author prefers to remain anonymous, except to say that he is a Somayajin (सोमयाजिन i.e. one who is authorized to perform the Soma sacrifice), belonging to the Nūtana griha or Navina Vipina or Pudumana family, and born at Sivapura (Trichur). Raja Raja Varma* opines that the date of the work is between 1375 and 1475 A.D. From a verse taken from Govinda Bhatta's Ganika Sūchika Grantha (गणिकसूचिकाग्रंथ), the date of *Karana Paddhati* is believed to be 1353 Saka, or 1430 A.D. Whish however gives the date of *Karana Paddhati* as 1733 A.D.

104. The series

$$\tan^{-1} t = t - \frac{t^3}{3} + \frac{t^5}{5} - \dots, \quad (t \leqslant 1)$$

is known as Gregory's series. James Gregory (1638—75) of Scotland obtained this result in 1671, probably by integrating in series the expression for $(1+t^2)^{-1}$. If we accept the date of *Karana Paddhati* as 1430, it will mean that this series and a number of its transformations had been obtained in India more than 200 years before Gregory. The *Tantra Sangraha* too gives particular cases of the series corresponding to $t=1$ and $t=1/\sqrt{3}$, and goes very much further by giving remarkably good rational approximations to π, and series which converge quite rapidly. These will be discussed below.

* *History af Sanskrit Literature in Kerala*, Vol. I, published in Malayalam by the Kamalalaya Book Depot, Trivandrum.

The *Yukti-Bhāsa* is definitely a work later than the above two but is of great importance in that it is the only work which gives the proofs of the theorems that it states. The information about its date and authorship is not only scanty, but confusing. The *Yukti-Bhāsa* appears to have been intended as a commentary of the *Tantra-Sangraha,* and according to one version was written by Nila Kantha himself. From a manuscript in the library of the Sanskrit college at Trippunnithura, it has been made out that the *Yukti-Bhāsa* was written in 1639 A.D. by one Brahmadatta. Whish gives the author's name as Kelallur Nambudri, but this may simply mean a Brahmin of the Nambudri sect, belonging to Kelallur village.

The *Sadratnamala* is a treatise on astronomy, written by Sankara Varman, who wrote this at the instance of Rāma Varman, brother of Uday Varman, king of Kerala. The date is uncertain ; two dates have been suggested : about 1530 A.D., or according to Whish 1832 A.D. The work itself is of lesser importance to us.

105. We shall now give the proof of Gregory's series, substantially as it is given in *Yukti-Bhāsa,* the alteration from $t=1$ as given there to a general value of $t \leqslant 1$ requiring only a slight and obvious alteration in the figure.

Lemma. Let O be the centre of a circle of unit radius, A any point on the circumference, and P, Q two points on the tangent at A. Let OP, OQ meet the circumference in p, q respectively, and let m be the foot of the perpendicular from p on OQ.

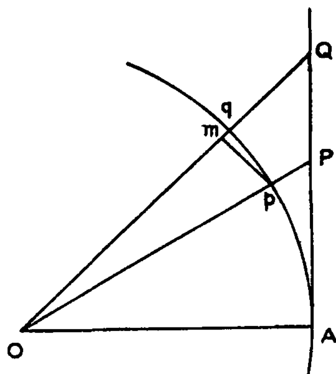


FIG. 22.

Then, (*i*) $pm = \dfrac{PQ}{OP.OQ}$

(*ii*) $arc\ pq = \dfrac{PQ}{1+AP^2}$.

The proof is elementary and straightforward.

Proof of Gregory's series. Let $\angle AOB \leqslant 45°$, B being any point on the tangent at A. Let $\tan \angle AOB = t \leqslant 1$. Divide AB into n equal parts at $P_0(=A)$, P_1, P_2, ..., $P_n(=B)$.

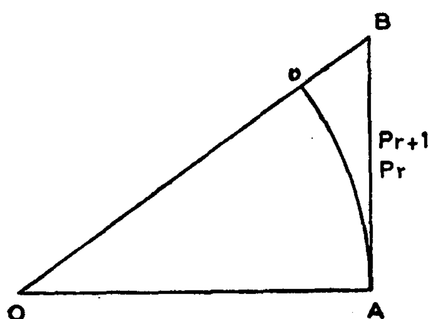


FIG. 23.

By the Lemma above,

$$\text{arc } Ab = \lim \sum_{r=0}^{n-1} \frac{P_r P_{r+1}}{1+AP_r^2} = \lim \sum_{r=0}^{n-1} \frac{t/n}{1+\left(\frac{rt}{n}\right)^2}$$

$$= \lim \sum_{r=0}^{n-1} \frac{t}{n}\left[1-\left(\frac{rt}{n}\right)^2 + \dots + (-1)^{\nu-1}\left(\frac{rt}{n}\right)^{2\nu-2} + \frac{(-1)^\nu (rt/n)^{2\nu}}{1+(rt/n)^2}\right],$$

by actual division.

Using the result

$$\lim_{n\to\infty} \frac{1^p+2^p+\dots+(n-1)^p}{n^{p+1}} = \frac{1}{p+1},$$

we get

$$\text{arc } Ab = t - \frac{t^3}{3} + \dots + (-1)^{\nu-1}\frac{t^{2\nu-1}}{2\nu-1} + (-1)^\nu R_\nu,$$

where $$0 < R_\nu < \frac{1}{2\nu+1}.$$

When $\nu \to \infty$, we get Gregory's series.

For the case $-1 \leqslant t < 0$, we have only to change the sign of t.

The case $t=1$ gives "Euler's series"

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \dots$$

which occurs both in *Karana Paddhati* and *Tantra Sangraha.* The case $t = \frac{1}{\sqrt{3}}$ gives

$$\pi = \sqrt{12}\left[1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \dots\right].$$

This occurs in *Tantra Sangraha.* In Europe, it was first given by Abraham Sharp (about 1717). B. B. Datta* states that it was first discovered by Bhadrambudhi Siddha Janma.

106. We now consider various transformations which transform the slowly convergent "Euler's series" to rapidly convergent series. We also obtain some rational approximations to π. The method is exactly as is set forth in *Yukti-Bhāsa*, though the details of the proof have been filled up by using modern analysis.

Grouping the terms of the series

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots \qquad (1)$$

two by two, we have

$$\frac{\pi}{4}=(1-\tfrac{1}{3})+(\tfrac{1}{5}-\tfrac{1}{7})+(\tfrac{1}{9}-\tfrac{1}{11})+\cdots$$

and

$$\frac{\pi}{4}=1-(\tfrac{1}{3}-\tfrac{1}{5})-(\tfrac{1}{7}-\tfrac{1}{9})-\cdots.$$

Hence,

$$\frac{\pi}{8}=\frac{1}{2^2-1}+\frac{1}{6^2-1}+\frac{1}{10^2-1}+\cdots \qquad (2)\ [T]\dagger$$

$$=\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\cdots \qquad (2')$$

and

$$\frac{4-\pi}{8}=\frac{1}{4^2-1}+\frac{1}{8^2-1}+\frac{1}{12^2-1}+\cdots \qquad (3)\ [T]$$

$$=\frac{1}{3.5}+\frac{1}{7.9}+\frac{1}{11.13}+\cdots. \qquad (3')$$

By subtracting (3) from (2),

$$\frac{\pi-2}{4}=\frac{1}{2^2-1}-\frac{1}{4^2-1}+\frac{1}{6^2-1}-\cdots. \qquad (4)\ [T]$$

Now in the series (1), let $S\left(\frac{n+1}{2}\right)$ and $S\left(\frac{n-1}{2}\right)$ denote the sums of the first $2m+1$ and $2m$ terms respectively, where $n=4m+1$. These provide rational approximations to $\frac{\pi}{4}$ when n is large. Let us

* *Bulletin of the Mathematics Association,* University of Allahabad, Vols. I and II, 1927-29.

† The square bracket [T] indicates that the formula is given in *Tantra Sangraha.*

try to improve them by adding corrections $-f(n+1)$ and $f(n-1)$ respectively, where $f(x)$ is a function to be suitably chosen. We write the improved approximations, by writing

$$T\left(\frac{n+1}{2}\right)=S\left(\frac{n+1}{2}\right)-f(n+1) \tag{5}$$

$$T\left(\frac{n-1}{2}\right)=S\left(\frac{n-1}{2}\right)+f(n-1). \tag{6}$$

$T\left(\frac{n+1}{2}\right)$ will be the partial sum of a certain series whose general term u_n is given by $T\left(\frac{n+1}{2}\right)-T\left(\frac{n-1}{2}\right)$. Subtracting (6) from (5), we have

$$u_n=\frac{1}{n}-f(n+1)-f(n-1). \tag{7}$$

Changing successively n to $n-2$, $n-4$, ..., 3, multiplying alternately by 1 and -1, and adding, we obtain

$$-u_3+u_5-\ldots+u_n=-\frac{1}{3}+\frac{1}{5}-\ldots+\frac{1}{n}+f(2)-f(n+1).$$

If $f(n)\to 0$ as $n\to\infty$, we then have

$$\frac{\pi}{4}=1-f(2)-u_3+u_5-\ldots+u_n-\ldots. \tag{8}$$

Suppose $f(x)$ is assumed to be of the form

$$2f(x)=\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\ldots. \tag{9}$$

Expanding $f(n+1)$ and $f(n-1)$ by Taylor's Theorem, we have from (7)

$$\frac{1}{n}=u_n+2\left[f(n)+\frac{f''(n)}{2!}+\frac{f^{(iv)}(n)}{4!}+\ldots\right.$$
$$\left.+\frac{1}{(2p)!}\{f^{(2p)}(n+\theta_1)-f^{(2p)}(n-\theta_2)\}\right], \tag{10}$$

where $0<\theta_1<1$, $0<\theta_2<1$.

Since $f(n)=O\left(\frac{1}{n}\right)$, $f''(n)$, $f^{(iv)}(n)$, etc., are $O\left(\frac{1}{n^3}\right)$, $O\left(\frac{1}{n^5}\right)$ etc. Hence if we propose that $u_n=O\left(\frac{1}{n^{2p+1}}\right)$, we obtain from (10),

$$\frac{1}{n}+O\left(\frac{1}{n^{2p+1}}\right)=2\left[f(n)+\frac{f''(n)}{2!}+\ldots+\frac{1}{(2p-2)!}f^{(2p-2)}(n)\right]. \tag{11}$$

Substituting (9) in (11), we can calculate $a_1, a_2, \ldots, a_{2p}$. We verify that $a_2 = a_4 = \ldots = a_{2p} = 0$. Any function of the type (9) can be taken as $f(x)$, after calculating $a_1, a_2, \ldots, a_{2p}$ in this way, the further coefficients a_{2p+1}, etc. being chosen arbitrarily in a convenient manner. We then use (7) and (8) to get a series for π in which $u_n = O\left(\frac{1}{n^{2p+1}}\right)$.

Putting $p=1$ in (11), we get $2f(n) = \frac{1}{n}$ and hence (5) gives the rational approximation

$$\frac{\pi}{4} \doteq 1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{n} \mp \frac{1}{2(n+1)}. \tag{12}$$

(7) gives $u_n = -\frac{1}{n(n^2-1)}$, and hence we obtain from (8),

$$\frac{\pi-3}{4} = \frac{1}{3^3-3} - \frac{1}{5^3-5} + \frac{1}{7^3-7} - \cdots \qquad (13)\ [T]$$

$$= \frac{1}{2.3.4} - \frac{1}{4.5.6} + \frac{1}{6.7.8} - \frac{1}{8.9.10} + \cdots \qquad (13')$$

Putting the terms on the right side of (13′) two by two in brackets, we obtain

$$\frac{\pi-3}{6} = \frac{1}{(2.2^2-1)^2-2^2} + \frac{1}{(2.4^2-1)^2-4^2} + \frac{1}{(2.6^2-1)^2-6^2} + \cdots \qquad (14)\ [K]$$

Next, putting $p=2$ in (11), we obtain from (9) and (11), $a_1 = 1$, $a_3 = -1$, $a_2 = a_4 = 0$.

Taking

$$2f(n) = \frac{1}{n} - \frac{1}{n^3} + \frac{1}{n^5} - \frac{1}{n^7} + \cdots$$

$$= \frac{n}{n^2+1},$$

we obtain from (5), the approximation

$$\frac{\pi}{5} \doteq 1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{n} \mp \frac{\frac{n+1}{2}}{(n+1)^2+1}. \qquad (15)\ [T]$$

(7) gives $\quad u_n = \frac{4}{n(\overline{n-1}^2+1)(\overline{n+1}^2+1)},$

after simplification. Hence, from (8),

$$\frac{\frac{\pi}{4}-\frac{4}{5}}{4}=-\frac{1}{3(2^2+1)(4^2+1)}+\frac{1}{5(4^2+1)(6^2+1)}-\dots$$

$$\text{or}\quad \frac{\pi}{16}=\frac{1}{1^5+4.1}-\frac{1}{3^5+4.3}+\frac{1}{5^5+4.5}-\dots \qquad (16)\ [T]$$

Next, putting $p=3$, we obtain as before from (9) and (11),

$$a_1=1,\ a_3=-1,\ a_5=5,\ a_2=a_4=a_6=0.$$

Taking

$$2f(n)=\frac{1}{n}-\frac{1}{n^3}+\frac{5}{n^5}-\frac{5^2}{n^7}+\dots$$

$$=\frac{n^2+4}{n(n^2+5)},$$

we obtain the rational approximation

$$\frac{\pi}{4}\doteqdot 1-\frac{1}{3}+\frac{1}{5}-\dots\pm\frac{1}{n}\mp\frac{\frac{(n+1)^2}{4}+1}{\left(\frac{n+1}{2}\right)\left\{(n+1)^2+4+1\right\}}. \qquad (17)\ [T]$$

Calculating u_n from (7), we finally obtain

$$\left(\frac{\pi}{4}-\frac{7}{9}\right)\Big/36=\frac{1}{(3^3-3)(2^2+5)(4^2+5)}-\frac{1}{(5^3-5)(4^2+5)(6^2+5)}+\dots \quad (18)$$

Karana Paddhati gives the value of π as $31415926536/10^{11}$, which is correct to 10 places of decimals. *Sadratnamāla* gives π as $314159265358979324/10^{18}$ correct to 17 places. These remarkable approximations must have been obtained from formulæ like (15) or (17). The labour involved is considerable. It has been verified by Prince Rāṃa Varma that $n=55$ gives π correct to 10 places from (17) and to 6 places only from (15).

107. We next apply a similar process to the series

$$\frac{\pi-2}{4}=\frac{1}{2^2-1}-\frac{1}{4^2-1}+\frac{1}{6^2-1}-\dots \qquad (4)$$

Let $n=4m+2$. Call the sum of n terms as $S(n)$. Let corrections $-f(n+1)$ and $f(n-1)$ be applied, so that

$$T\left(\frac{n}{2}\right)=S\left(\frac{n}{2}\right)-f(n+1)$$

$$T\left(\frac{n-2}{2}\right)=S\left(\frac{n-1}{2}\right)+f(n-1).$$

$\therefore$ If u_n is the nth term of a new series having $T\left(\frac{n}{2}\right)$ as the partial sum,

$$u_n = \frac{1}{n^2-1} - f(n+1) - f(n-1) \tag{19}$$

We choose $f(n)$ so that $u_n = O\left(\frac{1}{n^{2p+2}}\right)$. Since $f(n) = O\left(\frac{1}{n^2}\right)$, we take

$$2f(x) = \frac{1}{x}\left[\frac{a_1}{x} + \frac{a_2}{x^2} + \dots\right].$$

We obtain from (19),

$$\frac{1}{n^2-1} + O\left(\frac{1}{n^{2p+2}}\right) = \sum_{r=0}^{p-1} \frac{2}{(2r)!} f^{(2r)}(n). \tag{20}$$

We can calculate a_1, a_2, ..., a_{2p} as before, and we get $a_2 = a_4 = \dots = a_{2p} = 0$. $f(n)$ is indeterminate save for its first $2p$ coefficients.

In particular, when $p=1$, $2f(n) = \frac{1}{n^2}$. Hence

$$\frac{\pi-2}{4} \doteq \frac{1}{2^2-1} - \frac{1}{4^2-1} + \dots \pm \frac{1}{n^2-1} \mp \frac{1}{2(n+1)^2}. \tag{21}$$

Proceeding as previously, we can obtain from this value of $f(n)$ an alternating series in which $u_n = O\left(\frac{1}{n^4}\right)$.

When $p=2$, we obtain from (20), $a_1 = 1$, $a_3 = -2$, $a_2 = a_4 = 0$. We take

$$f(n) = \frac{1}{n^2}\left[1 - \frac{2}{n^2} + \left(\frac{2}{n^2}\right)^2 - \dots\right]$$

$$= \frac{1}{n^2+2}.$$

This gives

$$\frac{\pi-2}{4} \doteq \frac{1}{2^2-1} - \frac{1}{4^2-1} + \dots - \frac{1}{n^2-1} \mp \frac{1}{2(n+1^2+2)} \tag{22) [T]$$

This value of $f(n)$ will lead to an alternating series in which $u_n = O\left(\frac{1}{n^6}\right)$.

108. In earlier chapters we have remarked that the ancient Hindus were the first to realize the existence of irrational numbers, and to make arithmetical opera ions on them. Simple surd numbers were being used quite freely in the mathematics of the *Sulva Sutras*. The approximation $\pi = 3.1416$ was first given by Arya Bhata [499 A.D.]. Not only did he give this value, but he probably realised that π is an irrational number. A formal proof of this was given by Lambert in 1761, but Nila Kantha's remarks, dated about 1500 A.D., are quite interesting and of historical value. In his commentary of *Arya Bhatiya*, Nilakantha poses for himself the question, "why do we give for π an approximate value in place of the true value ?", and answers it by saying : "For this reason, the ratio of the circumference to the diameter can never be expressed as the ratio of two integers."

We quote the actual text from the *Aryabhatiya** of Arya bhatacarya with the Bhashya of Nila Kanthasomsutvan,

कुतः पुनर्वास्तवीं संख्यामुत्सृज्यासन्नेवेहोक्ता। उच्यते। तस्यावक्तुमशक्यत्वात्। कुतः। येनमानेन मीयमानो व्यासो निरवयवः स्यात् तेनैव मीयमानः परिधिः पुनः सावयव ऽवस्यात्। येनचमीयमानः परिधि निरवयवस्तेनैव मीयमानो व्यासोपि सावयव ऽव, इत्येकेनैव मानेन मीयमानयोरुभयोः क्वापि न निरवयत्वं स्यात्। महान्तमध्वानं गत्वाप्यल्पावयवत्वमेव लभ्यम्। निरवयवत्वंतु क्वापि न लभ्यमिति भावः।

Translation : Why then is it that discarding exact value, only the appoximate one has been mentioned here ? This is the answer : because it (the exact value) cannot be mentioned. If the diameter, measured with respect to a particular unit of measurement is commensurable, then w.r.t. the same unit, the circumference cannot be exactly measured ; and if w.r.t. any unit the circumference is commensurable, then w.r.t. the same unit, the diameter cannot be exactly measured. Thus there will never be commensurability for both w.r.t. the same unit of measurement. Even after going a long way, the degree of commensurability can be made very small, but absolute commensurability can never be attained.

* *Trivandrum Sanskrit Series*, No. 20, Part I, commentary on verse 10, Chapter II of *Aryabhatiya*.

109. The above discussion makes it quite certain that the series that are being called Gregory's series and Euler's series for π had been obtained and proved quite well in India long before the times of Gregory and Euler. More suitable names have to be proposed for these series to do justice to ancient India. All this work however raises a historical problem of first-rate importance. The nature of this work is so different from and the standard so much higher than the mathematics up to and including Bhaskara, that one is inclined to think that Indian mathematics after Bhaskara did not die out, but continued to flourish in some remote corner, at least, of the country. The *Yukti-Bhasa* and the *Tantra Sangraha* reveal to us the gulf which separates these works from those of their predecessors, like Bhaskara. The result

$$\lim_{n \to \infty} \frac{1^p + 2^p + \ldots + (n-1)^p}{n^{p+1}} = \frac{1}{p+1}$$

is not elementary, and its proof has not been indicated. It might have been obtained by reducing the limit of the sum to an integral. Western mathematics had not developed to such an extent as to make a claim that these Kerala mathematicians must have had some contact with the West. The date of the *Tantra Sangraha* is quite definite, and it is quite likely that the date of the *Yukti-Bhāsa* is not later than 1639 A.D. as stated above. All this leads us to believe that some mathematical work of a high order must have existed between 1150 A.D. and 1500 A.D. which was available to Nilakantha and others, and which is yet to be unearthed.

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